Consistent Variance Curve Models

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Consistent Variance Curve Models

Outline

- Introduction
  - Options on realized variance
- Variance Curve Models
  - General theory
  - Finite-dimensionally parameterized curves
  - Variance Curves in a Hilbert space
- Examples
- Hedging

Realized Variance

Trading volatility

Consistent Variance Curve Models

Introduction

- We are given an equity index (S&P, EuroSTOXX,…).
- Listed options and complex OTC products are traded on S.
  - "Volatility" drives the price of such options.
  - Can we also *trade* "volatilty" ?

- Well, we can trade *realized variance*.
- It is typically computed over the business days $0=t_0<\ldots<t_N=T$ using the estimator

$$V^N(T) := \sum_{i=1}^{N} \left(\log S_{t_i} - \log S_{t_{i-1}}\right)^2$$

(up to scaling). But we will assume that it is actually given as

$$V^N(T) \approx \langle \log S \rangle_T$$

Consistent Variance Curve Models

Introduction

- We will assume that S is continuous, that it pays no dividends and that the interest rates are zero. Hence, we may write it as

$$\begin{aligned} S_t &= \exp\left(X_t - \tfrac{1}{2}\langle X\rangle_t\right) \\ dX_t &= \sqrt{z_t}\, dB_t \end{aligned}$$

  on a stochastic base $(W, \mathrm{P}, \mathrm{F})$

  - The one-dimensional Brownian motion B is adapted to the filtration F.
  - The *short variance* process z is a predictable, integrable and non-negative.

- Realized variance is then the non-negative quantity

$$\langle \log S\rangle_T = \int_0^T z_s\, ds$$

Realized Variance

Variance Swaps

- The simplest product on realized variance is a *variance swap*.
- A variance swap is just a forward on realized variance:
  - At maturity T it pays the realized variance occurred during the life of the contract (usually in exchange for a previously agreed fixed strike K).
  - Such contracts are today liquidly traded on most major indices. In particular, their price processes are martingales under each equivalent martingale measure on $(\Omega, \mathcal{P}, \mathcal{F})$.
    - Let us assume that P is a martingale measure.
  - Then, the price $V_t(T)$ of a variance swap is just the expectation of the realized variance,

$$V_t(T) = \mathrm{E}\left[\int_0^T z_s \, ds \mid \mathcal{F}_t \right]$$

Realized Variance

Variance Swaps

- If European options are traded for all strikes, the price of a variance swap can in theory be computed in terms of European options using Neuberger's (1990) formula,

$$
\begin{aligned}
V_0(T) &= 2\,\mathrm{E}\left[-\int_0^T \sqrt{z_s}\, dB_s + \tfrac{1}{2}\int_0^T z_s\, ds\right] \\
&= 2\,\mathrm{E}\left[S_T - 1 - \log S_T\right] \\
&= 2\left\{\int_0^1 \frac{1}{K^2}\mathrm{Put}(T,K)\,dK + \int_1^\infty \frac{1}{K^2}\mathrm{Call}(T,K)\,dK\right\}
\end{aligned}
$$

  – This works only if option prices are available for all T.

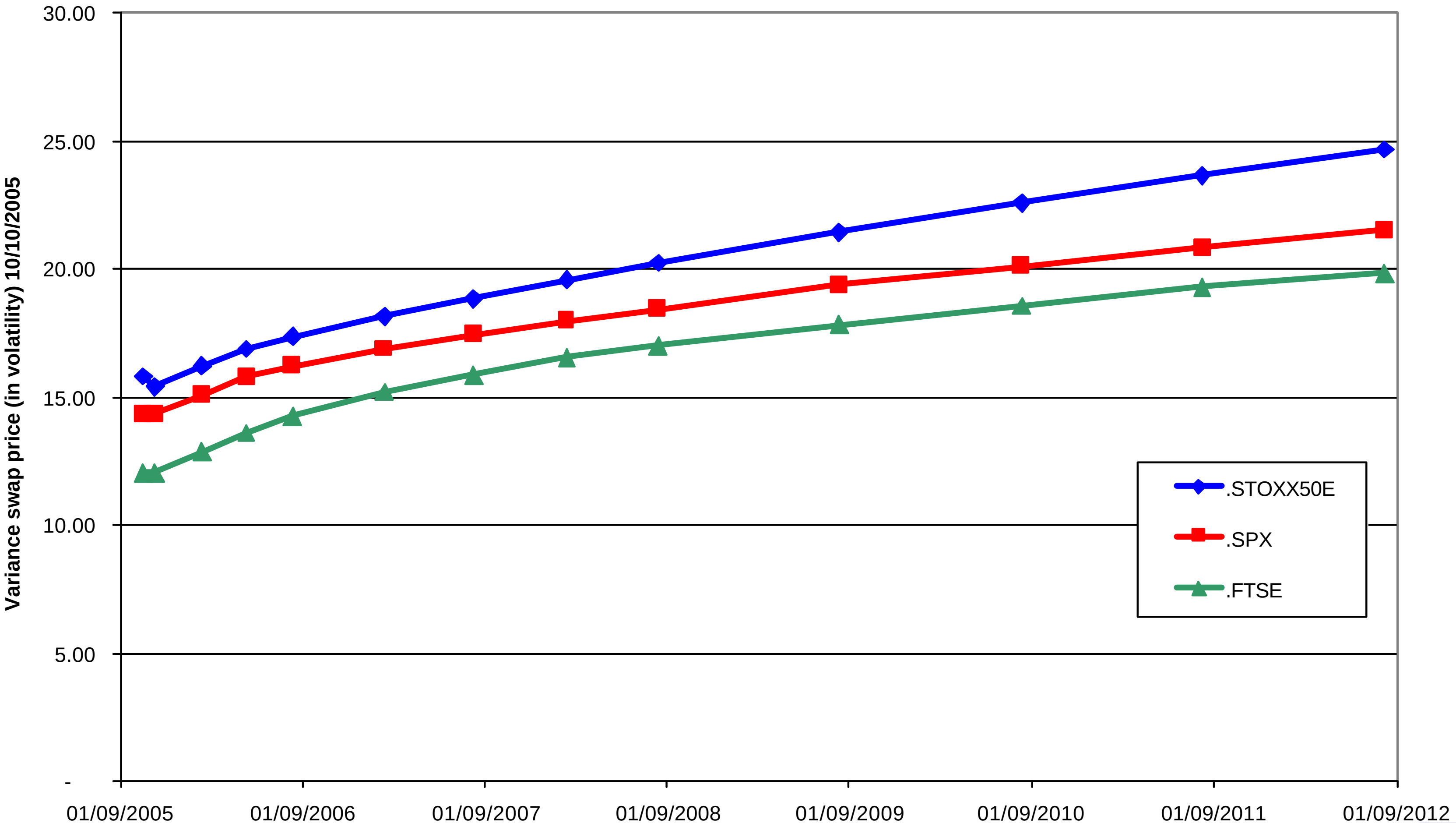
- The formula probably contributes to the fact that variance swaps are now liquidly traded.

  – An excellent reference is Demeterfi et al (1999).

Realized Variance

Variance Swaps

Prices are quoted in "volatility" $\sqrt{\frac{1}{T}\int_0^T z_s ds}$



Realized Variance

Beyond Variance Swaps

- Since variance swaps are liquidly traded, there is no need to price them.
- But what about more complex products:
  - *Calls* on realized variance

$$\left(\int_0^T z_s ds - K^2 \right)^+$$

  - *Volatility swaps*

$$\sqrt{\int_0^T z_s ds} - K$$

  - But also *forward started options*

$$\left(S_{T_2} / S_{T_1} - k \right)^+ = \left(\exp\left(\int_{T_1}^{T_2} \sqrt{z_s} dB_s - \tfrac{1}{2} \int_{T_1}^{T_2} z_s ds \right) - k \right)^+$$

Realized Variance

Beyond Variance Swaps

- The idea is to use the variance swaps themselves as underlying reference instruments.
  - A variance swap is a natural hedging instrument for such payoffs.
  - We can then attempt to hedge options on realized variance by delta-hedging with variance swaps.

- Mathematically, the term-structure of variance swaps reminds on the term-structure of discount bounds in interest rate models
  $\rightarrow$ *A term-structure problem is looming!*

Realized Variance

Forward Variance

- Variance swap prices are increasing with maturity T.
  - Their price at a later time t also depends on the past realized variance.
- To alleviate these unpleasant properties, note that

$$V_t(T) = \mathrm{E}\left[\int_0^T z_s ds \mid \mathcal{F}_t\right] = \int_0^T \mathrm{E}\left[z_s \mid \mathcal{F}_t\right] ds$$

can be differentiated in T to define the *forward variance*

$$v_t(T) := \partial_T V_t(T) = \mathrm{E}\left[z_T \mid \mathcal{F}_t\right]$$

Observation time (t)   **Maturity** (T)

- Note the similarity to the *forward rate* in interest rate theory.

Realized Variance

Modeling forward Variance

- Idea
  - Instead of starting with S, let us first specify the non-negative curve v.
  - Then, construct a (local) martingale S which has the correct quadratic variation.
  - The model shall also yield hedging ratios in terms of variance swaps.
  - The main difference is that forward variance can be <u>zero</u> for good economic reasons (holidays, suspended trading).
    - Short variance also becomes zero for standard stochastic volatility models such as Heston's.

Variance Curve Models

A structural approach

Variance Curve Models

Classic approach

- First, we focus on the classical setup.
  Assume we have a driving d-dimensional extremal Brownian motion W on the space $(\Omega, \mathrm{P}, \mathrm{F})$.

- <u>Definition</u>
  A family $v=(v(T))_{T \geq 0}$ is called a *Variance Curve Model* if

1. For each $T>0$, the process $v(T)=(v_t(T))_{t \in [0,T]}$ is a non-negative martingale:

$$dv_t(T) = \sum_{j=1,\dots,d} b_t^j(T) dW_t^j \qquad b^j(T) \in L^{\mathrm{loc}}$$

2. For each $T>0$, the initial variance swap prices are finite, i.e.

$$V_0(T) = \int_0^T v_0(s) ds < \infty$$

3. The curve $v_t(t)$ is left-continuous.

Variance Curve Models

Classic approach

- Properties
  - The price processes of variance swaps,

$$V_t(T) := \int_0^T v_t(s)ds$$

    are martingales.

  - The *short variance process*

$$z_t := v_t(t)$$

    is well defined, integrable and non-negative.

Variance Curve Models

Classic approach

- Properties
  Given any standard Brownian motion B on $(W,\mathrm{P},\mathrm{F})$, the process

$$dX_t = \sqrt{z_t}\, dB_t$$

  is a square-integrable martingale, so the via B *associated stock price*

$$S_t := \exp\left(X_t - \tfrac{1}{2}\langle X \rangle_t\right)$$

  is a local martingale.

  – B represents the *correlation structure* of S with v.

- Theorem
  For each variance curve model v and each Brownian motion B, the market

$$\left(S; (V(T))_{T\geq 0}\right)$$

  is free of arbitrage.

Variance Curve Models

Classic approach – Musiela-Parametrization

- As in interest rates, it is more convenient to work with fixed time-to-maturities $x{:=}T\text{-}t$. Hence we define the *Musiela parameterization*

$$\hat{v}_t(x) := v_t(t+x)$$

- Starting in Musiela-parametrization
  - Assume that $\sum_{j=1,\ldots,d} \int_0^\infty \int_t^\infty \partial_T b_t(T)^2 \, dT dt < \infty$
    Then,

$$d\hat{v}_t(x) := \partial_x \hat{v}_t(x) dt + \sum_{j=1,\ldots,d} \hat{b}_t^{\,j}(x) dW_t^{\,j}$$

defines a variance curve model in Musiela-parametrization.

Variance Curve Models

Classic approach – Fitting the market

- If v is represented as an exponential,

$$\hat{v}_t(x) := \exp(\hat{w}_t(x))$$

it allows us to fit the model easily to an observed market forward variance curve u_0 by setting $w_0 := \log u_0$.

  – This construction does not allow v to become zero.

- A more convenient approach is to use an existing curve v^{base} and set

$$\hat{v}_t(x) := \frac{u_0(t+x)}{\hat{v}_0^{\text{base}}(t+x)} \hat{v}_t^{\text{base}}(x)$$

Variance Curve Models

Variance Curve Functionals

- Problems with a specification with general integrands $b(T)$:
  - It is complicated to check whether v remains non-negative.
  - In practice, it is not clear how to handle such integrands computationally.
- Hence, we want to write

$$\hat{v}_t(x) := G(Z_t; x)$$

for some suitable non-negative function G and an m-dimensional Markov-process Z.

Variance Curve Models

Variance Curve Functionals

- Definition

1. A non-negative C[2,2]-function $G{:}DxR^{+}$ ® R^{+} is called a *Variance Curve Functional* if

$$\int_0^T G(z;x)dx < \infty$$

for all T and $z \in D$ where D is an open set in $R_{\geq 0}{}^{m}$.

2. We denote by Ξ the set of all $C=(m,s)$ for which the SDE

$$dZ_t = m(Z_t)dt + \sum_{j=1,\ldots,d} s^{j}(Z_t)dW_t^{j}$$

starting at any point $Z_0 \in D$ has a unique solution Z which stays in D.

Variance Curve Models

Variance Curve Functionals

- <u>Definition</u>
  We call $C=(m,s)\in\Xi$ a *consistent factor model* for G if for any $Z_0 \in D$,

$$\hat{v}_t(x) := G(Z_t; x)$$

  defines a variance curve model.

- <u>Theorem</u>
  This is the case if and only if Z stays in D and if

$$\partial_x G(z;x) = m(z)\partial_z G(z;x) + \frac{1}{2} s^2(z)\,\partial_{zz}^2 G(z;x)$$

  holds such that $G(Z_t;T\text{-}t)$ is a true martingale.

Variance Curve Models

Variance Curve Functionals

- Remarks
  - For each consistent pair (G,C), we obviously have

$$G(z;x) \equiv \mathrm{E}\left[G(Z_x;0) \mid Z_0 = z\right]$$

  - But our interest was asked the reverse question:
    given G, find a consistent $C=(m,s)\in \Xi$.

- The next logical step is to model the curve v as a process with values in a Hilbert space H.
  - We follow the path laid by Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004) and Teichmann (2005).

Variance Curve Models

Term-structure approach

- The main difference between variance curve and forward curves is that the curves v must remain non-negative (but *can* become zero).
  - The problem is that the “non-negative cone” is a very small set. Indeed it has no interior points.
  - However, if $G(D)$ is a sub-manifold with boundary of H, then it is sufficient to check whether v stays in $G(D)$. In this case we say $G(D)$ is *locally invariant* for v.
  - If G is moreover invertible, we can also directly construct a (locally) consistent factor model $C=(m,s)$ for G.

Variance Curve Models

Term-structure approach

- Assume that the variance curve v is given as a solution in H to

$$d\hat{v}_t = \partial_x \hat{v}_t dt + \sum_{j=1,\ldots,d} \hat{b}^j(\hat{v}_t) dW_t^j$$

where the coefficients b are locally Lipschitz vector fields.

- The Stratonovic-drift for v is as usual

$$b^0(\hat{v}) := \partial_x \hat{v} - \sum_{j=1,\ldots,d} Db^j(\hat{v}) \cdot b^j(\hat{v})$$

such that

$$d\hat{v}_t = \hat{b}^0(\hat{v}_t) dt + \sum_{j=1,\ldots,d} \hat{b}^j(\hat{v}_t) \circ dW_t^j$$

Variance Curve Models

Term-structure approach

- Theorem (Filipovic/Teichmann 2004)
  The sub-manifold $G(D)$ is locally invariant for v iff

1. We have $G(D) \subset \mathrm{dom}(\partial x)$,
2. In the interior of $G(D)$, we have

$$\hat{b}^j(\hat{v}) \in T_{\hat{v}} G(D) \qquad j = 0, \ldots, d$$

3. On the boundary $\partial G(D)$,

$$\hat{b}^0(\hat{v}) \in (T_{\hat{v}} G(D))_{\geq 0}$$

$$\hat{b}^j(\hat{v}) \in T_{\hat{v}} \partial G(D) \qquad j = 1, \ldots, d$$

holds.

Variance Curve Models

Term-structure approach

- If we can invert G, then $C=(m,s)$ with

$$s^j(z) := \partial_z G^{-1}(\hat{b}^j(G(z))$$

$$m(z) := \partial_z G^{-1}(\hat{b}^0(G(z)) + \sum_{j=1,..d} (\partial_z s^j)(z) s^j(z)$$

is (locally) a consistent factor model for G.

Back to reality: A Simple Example

Linear mean-reversion

Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- Example
  A very basic example is the “linearly mean-reverting” functional:

„Speed of mean-reversion“

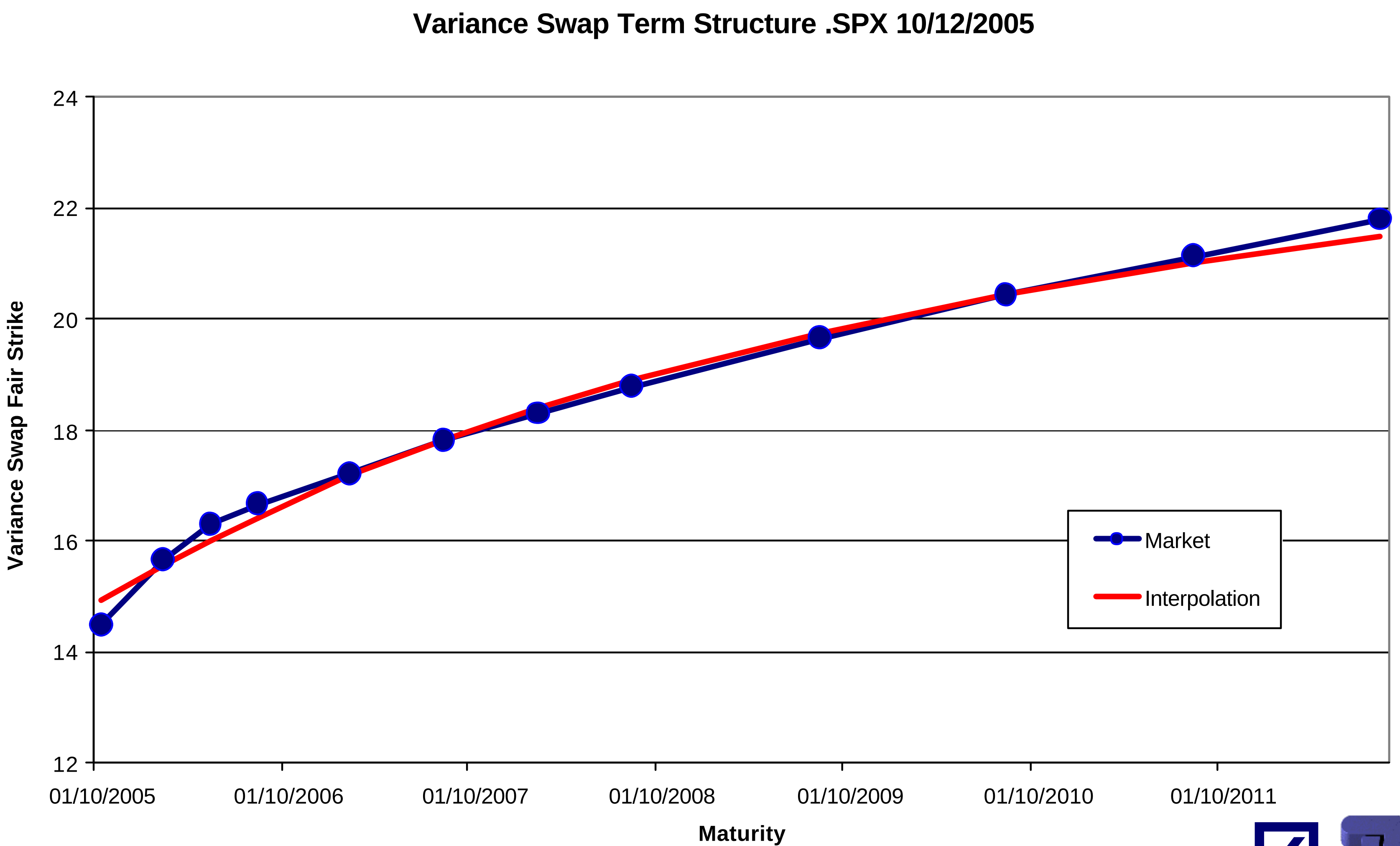
$$G(z;x) = z_2 + (z_1 - z_2)e^{-z_3 x}$$

„Long variance“ „Short variance“

for $z_1 \geq 0$ and $z_2, z_3 > 0$.

Variance Curve Models

Variance Curve Functionals – Linear mean-reversion



Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- Question: What dynamics can a consistent process $Z=(Z_1, Z_2, Z_3)$ have?
- The coefficients m and s have to satisfy

$$\partial_x G(z;x) = m(z)\partial_z G(z;x) + \frac{1}{2} s^2(z)\partial^2_{zz} G(z;x)$$

1. First, we see that

$$\partial^2_{z_3 z_3} G(z;x) = (z_1 - z_2) x^2 e^{-z_3 x}$$

   Since no term $x^2 e^x$ appears on the left hand side, we must have $s_3=0$.

2. The same line of thought applied to

$$\partial_{z_3} G(z,x) = -(z_1 - z_2) x e^{-z_3 x}$$

   shows that we also have $m_3=0$.

Hence, the speed of mean-reversion cannot be stochastic.

Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- For the other two parameters, we find that while s is unconstrained,

$$m_2(z) = 0$$
$$m_1(z) = z_3(z_2 - z_1)$$

In other words: The only consistent processes for this choice of G are of Heston-type

Linear mean-reversion drift

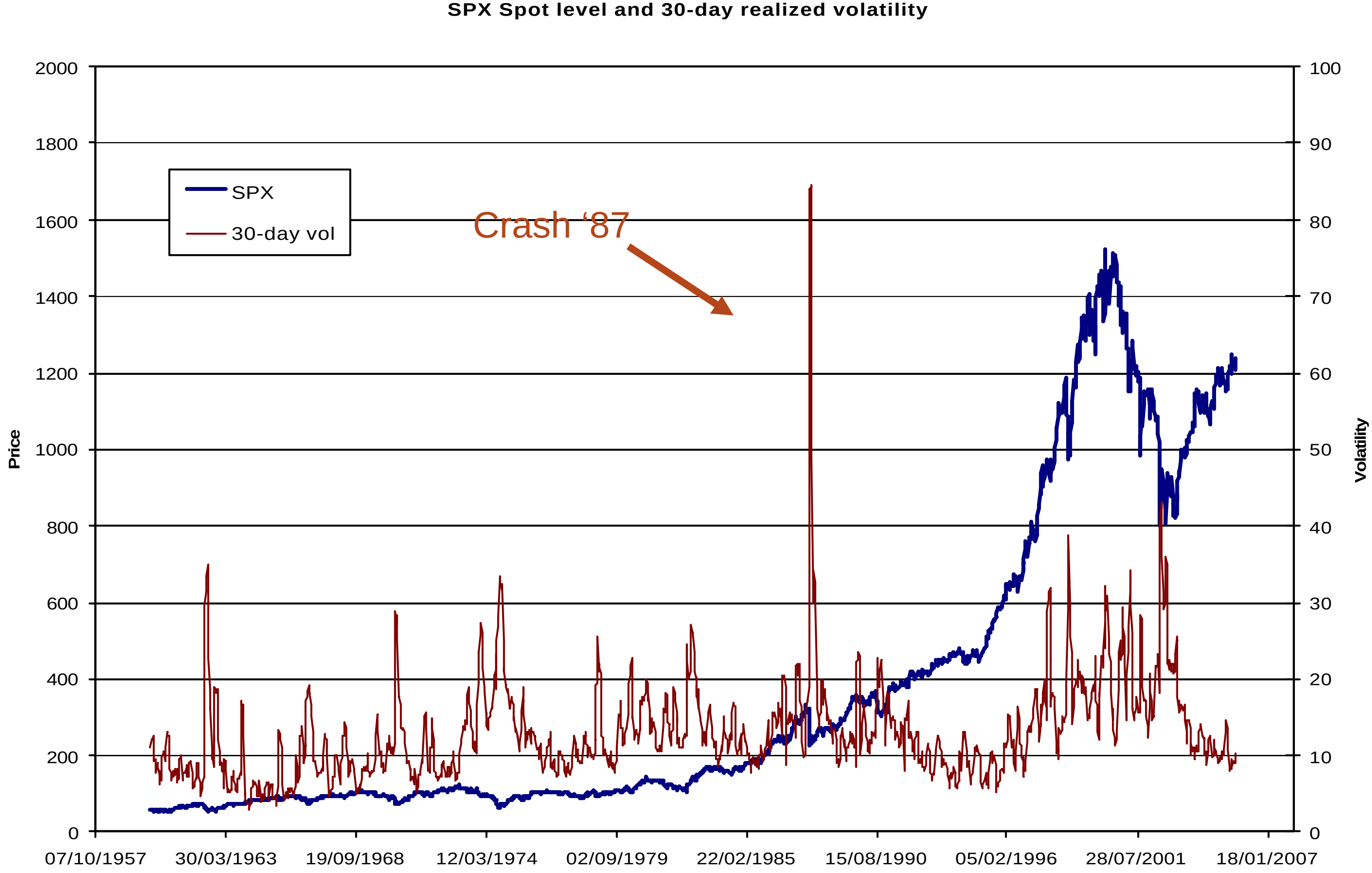
$$dz_t = k(q_t - z_t)dt + s_1(z_t, q_t)dW_t$$
$$dq_t = s_2(z_t, q_t)dW_t$$

VolOfVol can freely be chosen as long as z remains non-negative.

Mean-reversion level q is a positive martingale.

Variance Curve Models

Why mean-reversion?

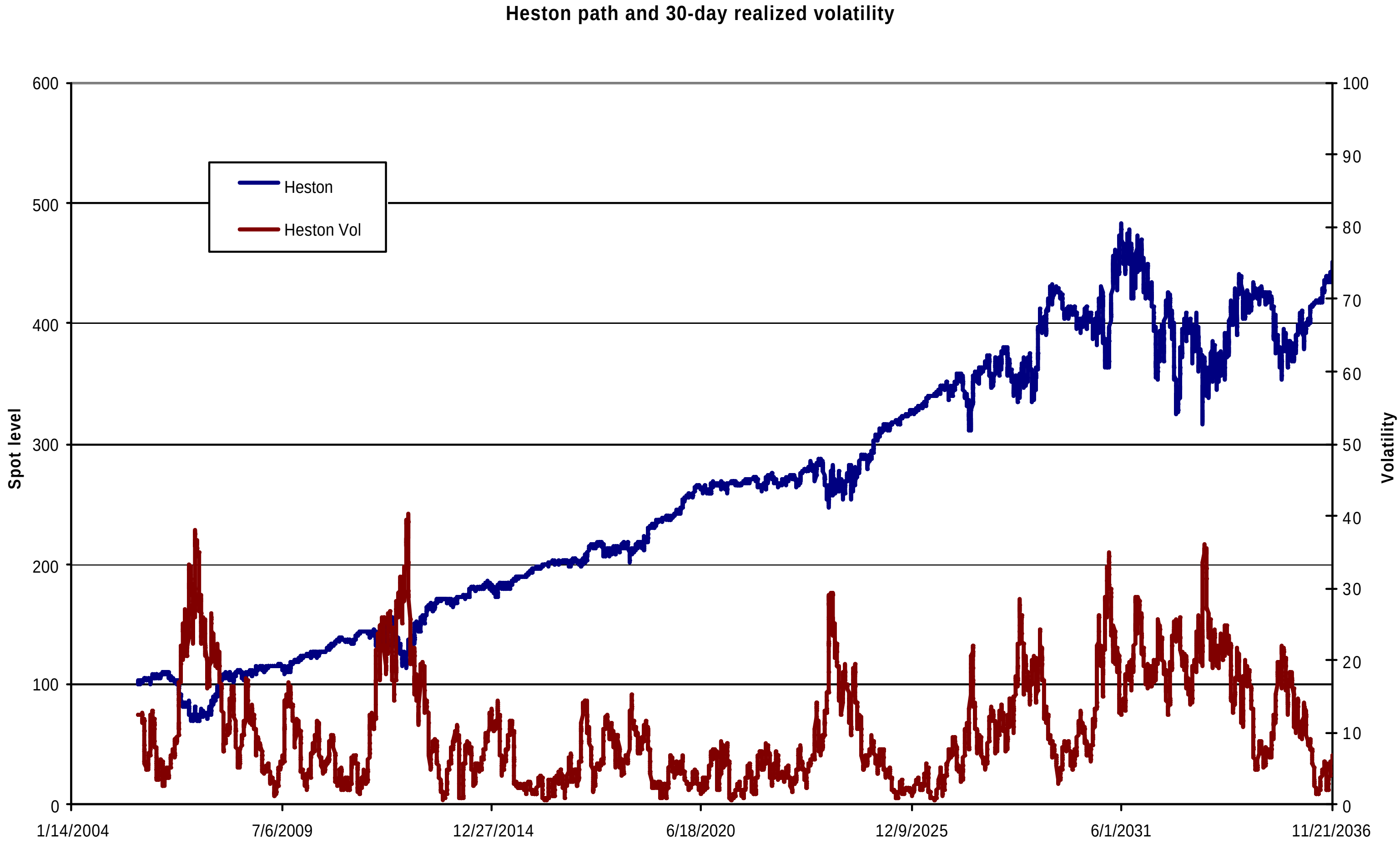


Variance Curve Models

Why mean-reversion?

Unconstrained Calibration

ShortVol	14.4%
LongVol	28.7%
RevSpeed	0.23
Correlation	-0.74
VolOfVol	26.3%



Variance Curve Models

Variance Curve Functionals

- Proposition
  The observation that mean-reversion speeds must be constant holds for all polynomial-exponential functionals, i.e. if

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\sum_{i=1}^{n} p_i(z;x)e^{-z_i x}$$

  (where $(p_i)_i$ are polynomials), then the first n components must be constant (cf. Filipovic 2001 for interest rates).

- A similarly restrictive result can be shown for functionals of the form

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\exp\left\{\sum_{i=1}^{n} p_i(z;x)e^{-z_i x}\right\}$$

  – The parameters in the exponent come in pairs, where one is twice as large as the other (again Filipovic 2001).

Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

- Another example of the polynomial-exponential class is

$$G(z;x) = z_3 + (z_1 - z_2)e^{-kx} + (z_2 - z_3)\frac{k}{k-c}\left(e^{-cx} - e^{-kx}\right)$$

  - A consistent factor model for this G must have the form

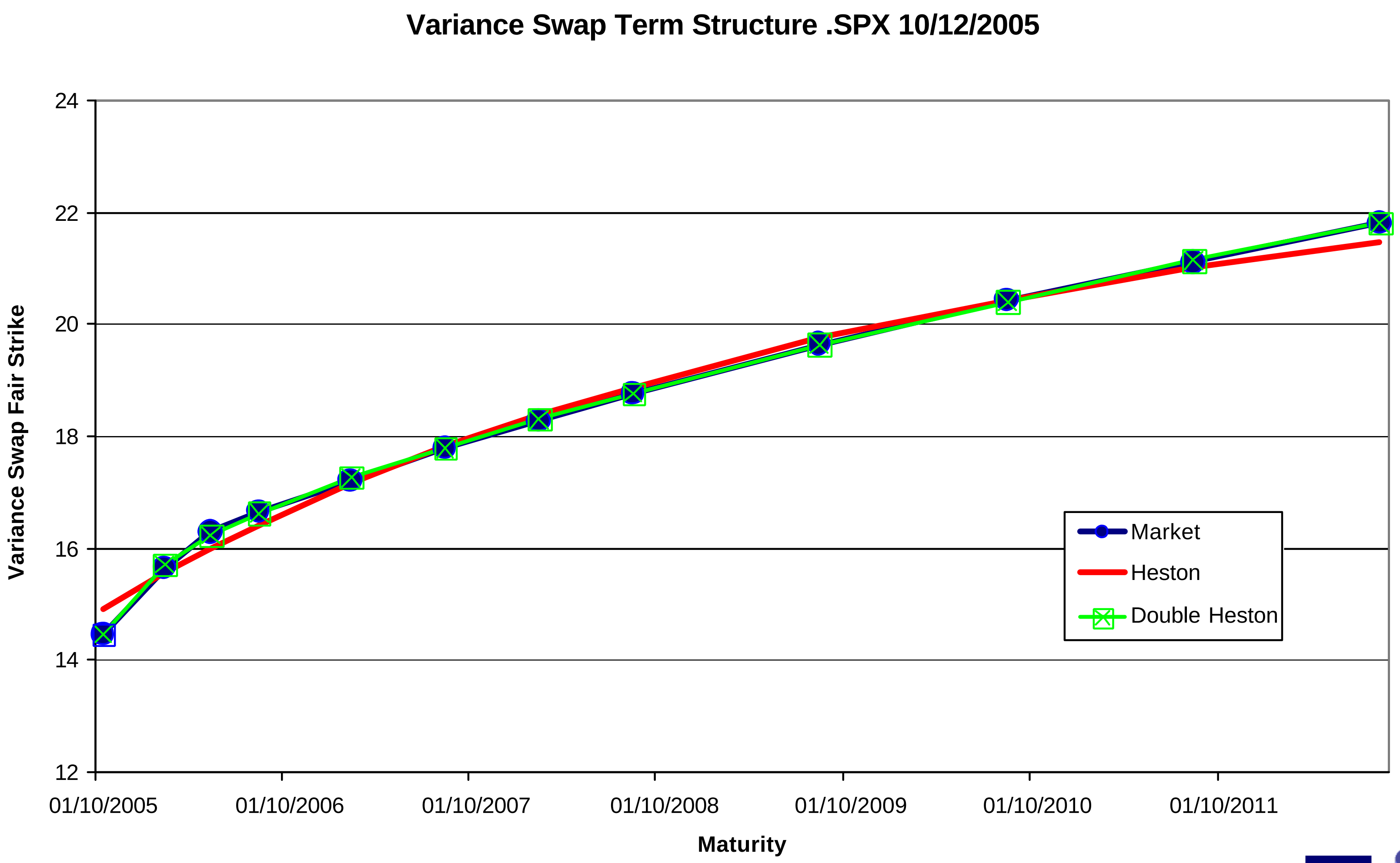
$$\begin{aligned} dZ_t^1 &= k(Z_t^2 - Z_t^1)dt + s_1(Z_t)dW_t \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + s_2(Z_t)dW_t \\ dZ_t^3 &= s_3(Z_t)dW_t \end{aligned}$$

    which we call “double mean-reverting”.

  - Quite a good fit for most indices (at least during the course of the last year).

Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion



Hedging

Using Variance Curve Models to hedge Options on Variance

Hedging

How to hedge with variance curve models

- We are back in our initial classical setting, i.e. we have decided to use a consistent variance curve model,

$$\hat{v}_t(x) = G(Z_t; x)$$

$$dZ_t = m(Z_t)dt + \sum_{j=1,\dots,d} s^{j}(Z_t)dW_t^{j}$$

$$z_t := \hat{v}_t(0)$$

- We want to price and hedge an “option on realized variance” with (bounded) European payoff h. Its price process is given as

$$\mathrm{E}\left[h\left(\int_0^T z_s ds \right) \middle| F_t \right]$$

Hedging

How to hedge with variance curve models

- Define the *variance swap price* function

$$\overline{G}(z;x) := \int_0^x G(z;y)\,dy$$

and assume that there exist constant $0<e<t_1<\cdots<t_m$ such that

$$\overline{G}_{t_1,\ldots,t_m}(z) := (\overline{G}(z;t_1),\ldots,\overline{G}(z;t_m))$$

is invertible for all $t_k := t_k - t$ for $0 \le \tau \le \varepsilon$.

- This then allows to recover Z in any interval $[a,b]$ by

$$Z_t = \overline{G}^{-1}_{T_1-t,\ldots,T_m-t}\big(V_t(T_1)-V_t(t),\ldots,V_t(T_m)-V_t(t)\big)$$

where $T_k := a + t_k$.

Variance swap

Running realized variance

Hedging

How to hedge with variance curve models

- Due to the Markov-property of Z, we have

Running realized variance

$$C_t(Z_t, V_t(t)) := \mathrm{E}\left[h\left(\int_0^T z_s ds \right) \middle| Z_t; V_t(t) \right]$$

- To hedge this payoff in the interval $[a,b]$, we can write it due to our assumptions on G as

$$C_t(Z_t, V_t(t)) \equiv C_t(V_t(T_1), \ldots, V_t(T_m), V_t(t))$$

such that (under the assumption that C is smooth enough)

$$dC_t(\cdots) = \sum_{k=1}^m \partial_{V_k} C_t(\cdots) dV_t(T_k)$$

Hedging

How to hedge with variance curve models

- This

$$dC_t(\cdots) = \sum_{k=1}^{m} \partial_{V_k} C_t(\cdots) dV_t(T_k)$$

is the desired hedge of h in terms of variance swaps.

  – For options on variance, this is a “natural” hedge.

  – It can also be used for standard options (a delta-term will appear).

  – For forward started options, correlation (skew) risk should be taken into account.

- In practise, the above “VarSwapDelta” hedging ratios are computed via bumping of the variance swap price.

Variance Curves

Future

Variance Curves

Future

- Challenges ahead
  - Incorporation of stochastic interest rates and dividends (in particular long-term deals could exhibit strong exposure to stochastic interest rates).
  - Jumps both in the underlying and the variance process (witness S&P return graph earlier).
  - Correlation between the variance curves between different underlyings (our suspicion is that it is actually quite high).
- The dream, however, is to characterize the entire implied volatility surface or, equivalently, the “forward probability distribution” of S.

Thank you very much for your attention.

Working paper and at http://www.math.tu-berlin.de/~buehler/

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