Delta-Hedging Works

Market Completeness for Factor Models on the example of Variance Curve Models

Conference on small time asymptotics, perturbation theory and heat kernel methods in mathematical finance

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Abstract

We discuss market completeness for diffusion-driven factor models beyond the classic requirement that the volatility matrix of traded instruments is invertible.

We show that the market generated by a finite-dimensional diffusion model is complete as soon as the coefficients of the SDE are $d\langle S\rangle \otimes dP$-almost surely C^1 with locally Lipschitz derivatives. As a consequence, when factor models are considered as diffusions in Hilbert spaces, then any such factor model which admits a finite dimensional representation creates a (locally) complete market.

This is illustrated on the example of Variance Swap Curve Market Models.

Delta-Hedging Works

on the example of Consistent Variance Curve Models

Outline

- Introduce generic Variance Swap Models
- On Hilbert Spaces
- Market Completeness

Realized Variance

Trading volatility

Realized Variance

Introduction

- *Realized variance* of a stock price process $S=(S_t)_t$ over business days $0=t_0<\cdots<t_n=T$ is defined as

$$\mathrm{RV}(t_0,t_n) := \sum_{i=1}^{n}\left(\log\frac{S_{t_t}}{S_{t_{t-1}}}\right)^2$$

- A *variance swap* pays out the annualized variance, i.e.

$$\frac{252}{n}\mathrm{RV}(t_0,t_n)$$

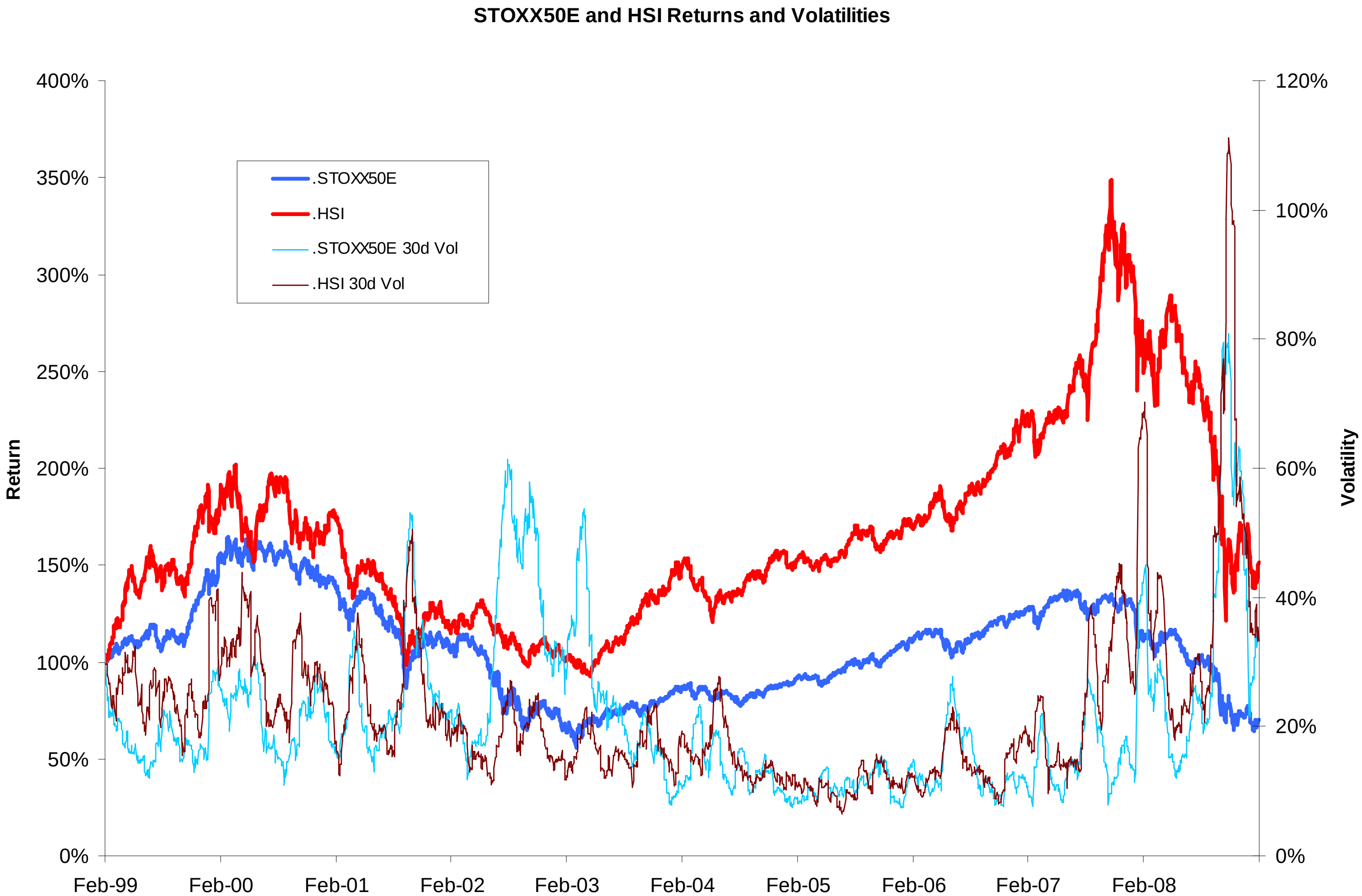
  – It is quoted in volatility but always traded in variance

  – We denote the (non-annualized) price as

$$V_{\mathrm{t}}(t_0,t_n) := \mathrm{E}_{\mathrm{t}}\left[\mathrm{RV}(t_0,t_n)\right]$$

Realized Variance
Introduction

$$t \mapsto \sqrt{\frac{252}{30}\mathrm{RV}(t, t+30d)}$$



Realized Variance

Introduction

- *Quadratic variation* is an unbiased estimator of realized variance, ie

$$\mathrm{E}\left[\langle \log S\rangle_T\right] \approx \mathrm{E}\left[\sum_{i=1}^{n}\left(\log S_{t_t} / S_{t_{t-1}}\right)^2\right]$$

(This is also true if S has a drift and potentially jumps)

  - Under any pricing measure

$$V_t(t_0, t_n) = \mathrm{E}\left[\langle \log S\rangle_{t_n} - \langle \log S\rangle_{t_0} \mid \mathrm{F}_t\right]$$

  - Prices are liquidly traded for indices (OTC).
  - Single stock variance swaps caused a lot of pain end of 2008

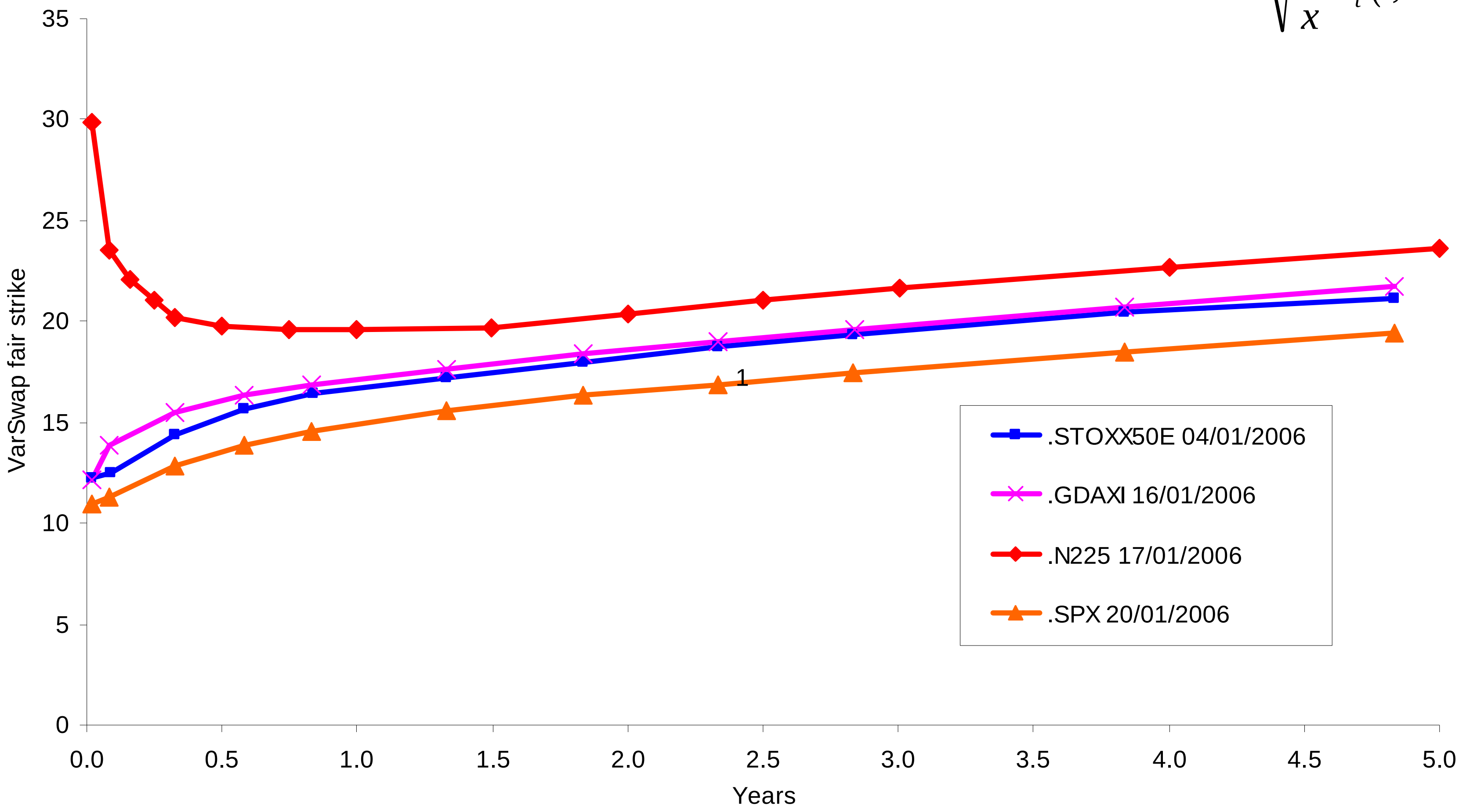
Realized Variance

Variance Swaps

Prices are quoted in "volatility"

$$t \mapsto \sqrt{\frac{1}{x} \mathrm{V}_t\left(t, t+x\right)}$$

Variance Swap Market Prices



Realized Variance
Beyond Variance Swaps

… but what about more complex products:

- *Straddles* on realized variance

$$\left| RV(0,T) - K^2 \right|$$

- *Volatility swaps*

$$\sqrt{RV(0,T)} - K$$

- To risk-manage such products the idea that variance swaps can be used to “delta-hedge” more complex options on realized variance.
  - For that, we obviously need a notion of completeness..

Variance Curve Models

Classic Approach

Variance Curve Models

Program

1. Instead of starting with S as in classic stochastic volatility models, let us first specify the *dynamics of the variance swaps*.
2. Then, construct a (local) martingale S which has the correct quadratic variation such that the market of variance swaps and stock is *free of arbitrage*.
3. The correlation between the Brownian motion which drives S and the variance curve will act as a *skew parameter*.
4. Since we are fundamentally aiming at *replication*, we provide criteria when the market is complete.
   - The latter excludes per se jumps in our discussion which are otherwise an important part of variance modelling.

Variance Curve Models

Classic approach

- Assume we have a driving d-dimensional extremal Brownian motion W on the space $(\Omega, \mathrm{P}, \mathrm{F})$.
- Recall that under any pricing measure

$$V_t(0,T) = \mathrm{E}_t\left[\langle \log S \rangle_T\right]$$

- The idea is to specify directly the dynamics of *instantaneous forward variance*

$$v_t(T) = \partial_T V_t(0,T)$$

very much like we specify the "forward rates" in HJM models:

$$f_t(T) \approx \partial_{T'}\big|_{T'=T} \log P(t,T),$$

Variance Curve Models

Classic approach

- Definition
  A family $v=(v(T))_{T\geq 0}$ is called a [local] *Variance Curve Model* if

1. For each $T>0$, the process $v(T)=(v_t(T))_{t\in[0,T]}$ is a non-negative [local] martingale:

$$dv_t(T)=\sum_{j=1,\ldots,d}\beta_t^j(T)dW_t^j \qquad \beta^j(T)\in L^{\mathrm{loc}}$$

Set of intregrable, predictable processes wrt W.

2. For each $T>0$, the initial variance swap prices are finite, i.e.

$$V_0(0,T)=\int_0^T v_0(s)ds<\infty$$

3. The curve $v_t(t)$ is left-continuous.

Variance Curve Models

Classic approach

- Properties
  - The price processes of variance swaps,

$$V_t(T_1,T_2) := \int_{T_1}^{T_2} v_t(s)ds$$

    are [local] martingales.

  - The *short variance process*

$$\zeta_t := v_t(t)$$

    is well defined, integrable and non-negative.

Variance Curve Models

Classic approach

- Properties
  Given any standard Brownian motion B on $(\Omega,\mathrm{P},\mathrm{F})$, the process

$$dX_t = \sqrt{\zeta_t}\, dB_t$$

is a square-integrable martingale, so the via B *associated stock price*

$$S_t := \exp\left(X_t - \tfrac{1}{2}\langle X \rangle_t\right)$$

is a local martingale.

  – B represents the *correlation structure* of S with v.

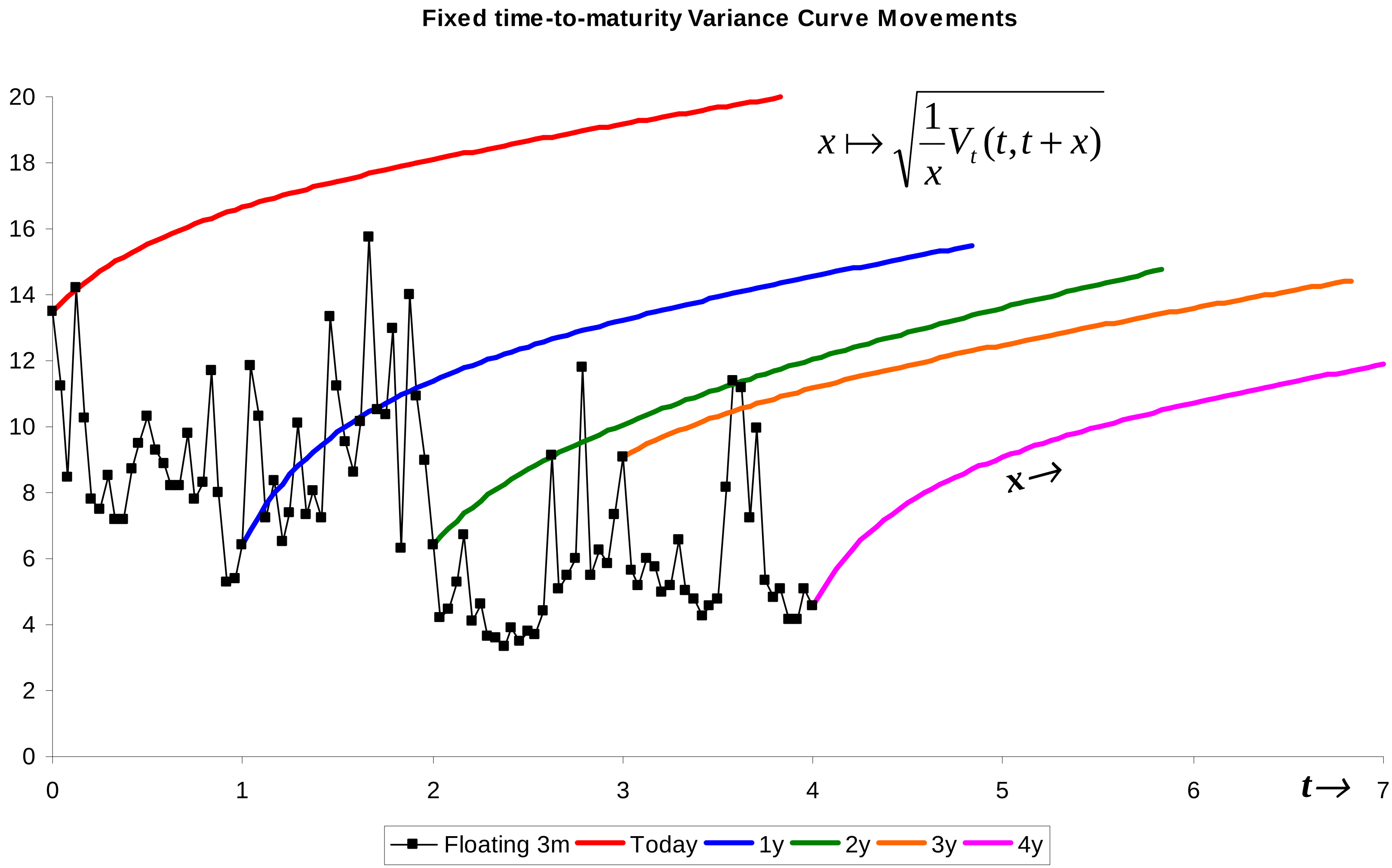
- Theorem
  For each variance curve model v and each Brownian motion B, the market

$$\left(S; (V(T_1,T_2))_{T_2>T_1\geq 0}\right)$$

is free of arbitrage.

Variance Curve Models

Classic approach –Musiela parametrization



Variance Curve Models

Classic approach – Musiela-Parametrization

- As in interest rates, it is more convenient to work with fixed time-to-maturities $x:=T-t$. Hence we define the *Musiela parameterization*

$$u_t(x) := v_t(t+x) \qquad\qquad v_t(T) = u_t(T-t)$$

- Starting in Musiela-parametrization
  – Assume that $\sum_{j=1,\dots,d} \int_0^\infty \int_t^\infty \partial_T \beta_t(T)^2 \, dT dt < \infty$
  Then,

$$du_t(x) := \partial_x u_t(x)dt + \sum_{j=1,\dots,d} b_t^j(x) dW_t^j$$

defines a local variance curve model in Musiela-parametrization.

Variance Curve Models

Classic approach – step one

- The previous discussion shows that it is remarkably easy to construct an arbitrage-free market with Variance Curve Models.
- Problems are

1. The general predictable integrands are far too general
    1. It is difficult to check whether v remains non-negative.
    2. Numerically intractable.
2. We would much prefer a representation in terms of a “driving” *finite-dimensional Markov process* to actually be able to implement the model on a computer.

3. Not discussed today: how do I fit an initial term structure from the market perfectly (cf. HJM models) → see http://www.math.tu-berlin.de/~buehler

Variance Curve Models

Consistency

- Ideally, we want to write

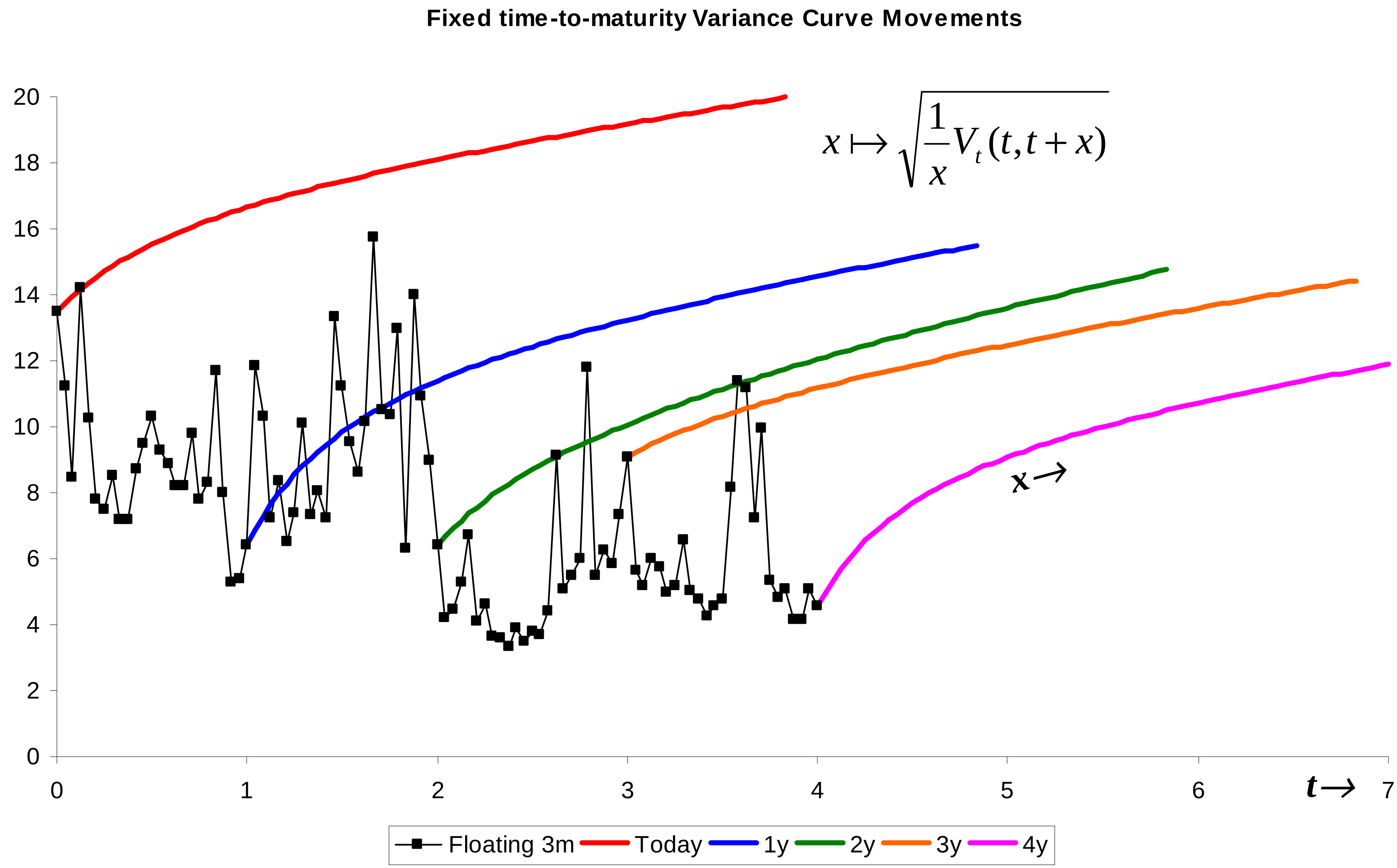
$$u_t(x) := G(Z_t; x)$$

- for some suitable non-negative function G and an m-dimensional Markov-process Z.
  - The function G is the "interpolation function" for the forward variances.

- Key point:
  We *first* chose the function G and then try to find the space of "suitable" parameter processes Z.

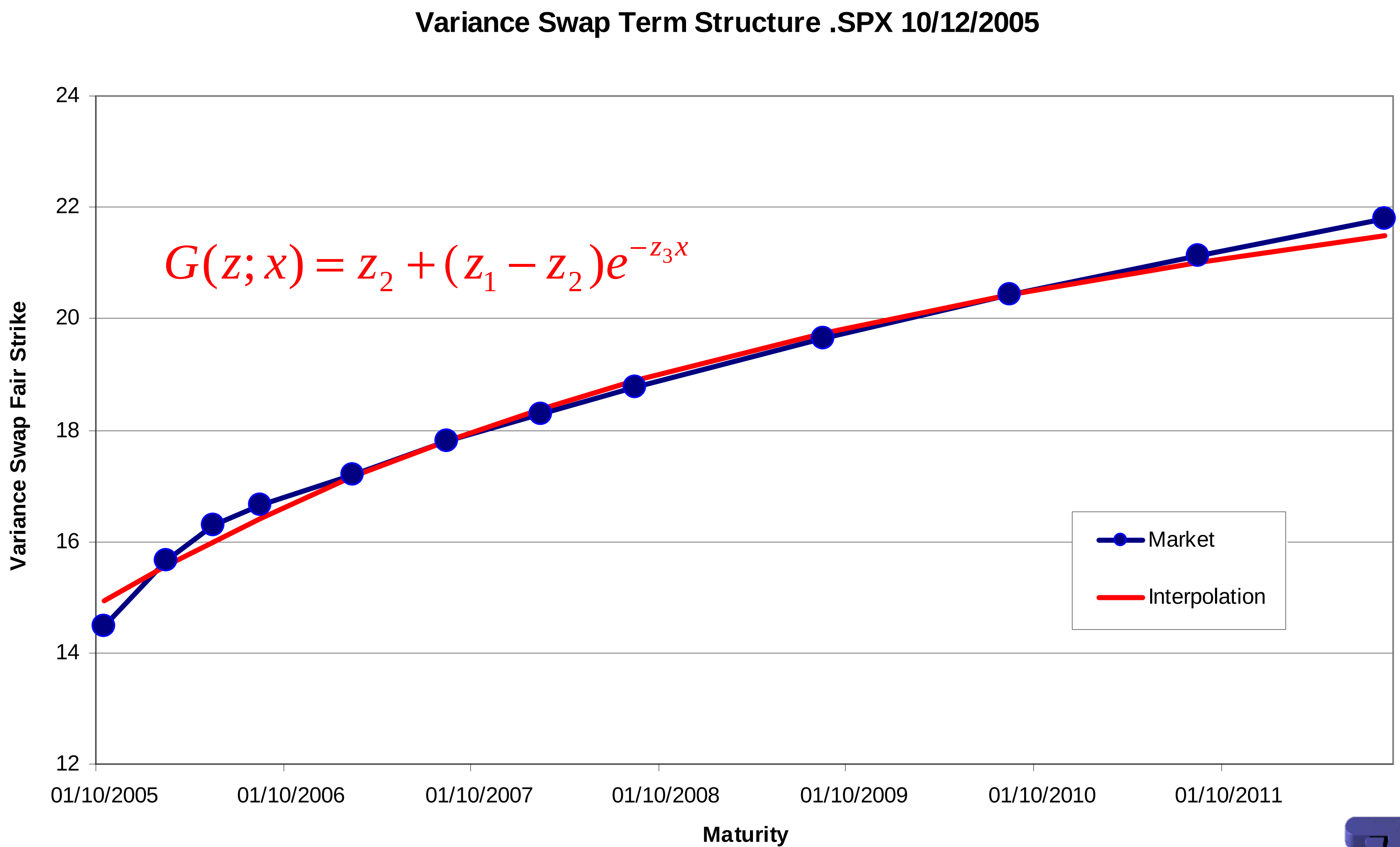
Variance Curve Models

Consistency



Variance Curve Models

Consistency



Variance Curve Models

Consistency

- <u>Definition</u>

1. A non-negative $C^{2,2}$-function $G{:}DxR^{+} \to R^{+}$ is called a *Variance Curve Functional* if

$$\int_0^T G(z;x)dx < \infty$$

   for all T and $z \in D$ where D is an open set in $R_{\geq 0}{}^{m}$.

2. We denote by Ξ the set of all $C=(\mu,\sigma)$ for which the SDE

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\ldots,d} \sigma^j(Z_t)dW_t^j$$

   starting at any point $Z_0 \in D$ has a unique solution Z which stays in D.

   – Time-dependency is included in this setup.

Variance Curve Models

Consistency

- Definition
  We call $C=(\mu,\sigma)\in\Xi$ a *consistent factor model* for G if for any $Z_0\in D$,

$$u_t(x) := G(Z_t; x)$$

  defines a local variance curve model.

- Theorem
  This is the case if and only if Z stays in D and if

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^T\sigma(z)\,\partial^2_{zz}G(z;x)$$

  holds.
  - Note that we are given G and look for $C=(\mu,\sigma)\in\Xi$ contrary to common applications.

Variance Curve Models

Consistency

- <u>Local Correlation and the Markov property</u>
  Given a consistent factor model $C=(\mu,\sigma)\in\Xi$ and a "correlation function" $\rho:R^{+}xD\rightarrow[-1,1]^{d}$ with $|\rho|=1$, we can always define

$$dS_t = \sum_{j=1,\dots,d} S_t \rho^j(S_t;Z_t)\left\{\sqrt{G(Z_t;0)}dW_t^j\right\}$$

  such that the process (S,Z) is Markov and S is a local martingale (note that the SDE does not explode).

  – The Markov property is essential for market completeness as we will see later.

  – Local-Stochastic volatility "mixture models" are also part of this framework

Variance Curve Models

Consistency – Examples

- <u>Example</u>
  A very basic example is the “linearly mean-reverting” functional:

„Speed of mean-reversion“

$$G(z;x) = z_2 + (z_1 - z_2)e^{-z_3 x}$$

„Long variance“ „Short variance“

for $z_1 \geq 0$ and $z_2, z_3 > 0$.

Variance Curve Models

Consistency – Linear mean-reversion

- For the other two parameters, we find that while σ is unconstrained,

$$\mu_2(z) = 0$$
$$\mu_1(z) = z_3(z_2 - z_1)$$

In other words: the only consistent processes for this choice of G are of Heston-type

Linear mean-reversion drift

$$\begin{aligned} d\zeta_t &= \kappa(\theta_t - \zeta_t)dt + \sigma_1(\zeta_t, \theta_t)dW_t \\ d\theta_t &= \sigma_2(\zeta_t, \theta_t)dW_t \end{aligned}$$

VolOfVol can freely be chosen as long as ζ remains non-negative.

Mean-reversion level θ is a positive martingale.

Variance Curve Models
Consistency

- Proposition
  The observation that mean-reversion speeds must be constant holds for all polynomial-exponential functionals, i.e. if $(p_i)_i$ are polynomials

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\sum_{i=1}^{n} p_i(z;x)e^{-z_i x}$$

  then the first n components must be constant (cf. Filipovic 2001 for interest rates).

- A similarly restrictive result can be shown for functionals of the form

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\exp\left\{\sum_{i=1}^{n} p_i(z;x)e^{-z_i x}\right\}$$

  - The parameters in the exponent come in pairs, where one is twice as large as the other (again Filipovic 2001).

Variance Curve Models

Consistency – Example linear mean-reversion

- Another example of the polynomial-exponential class is

$$G(z;x) = z_3 + (z_1 - z_2)e^{-\kappa x} + (z_2 - z_3)\frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right)$$

  – A consistent factor model for this G must have the form

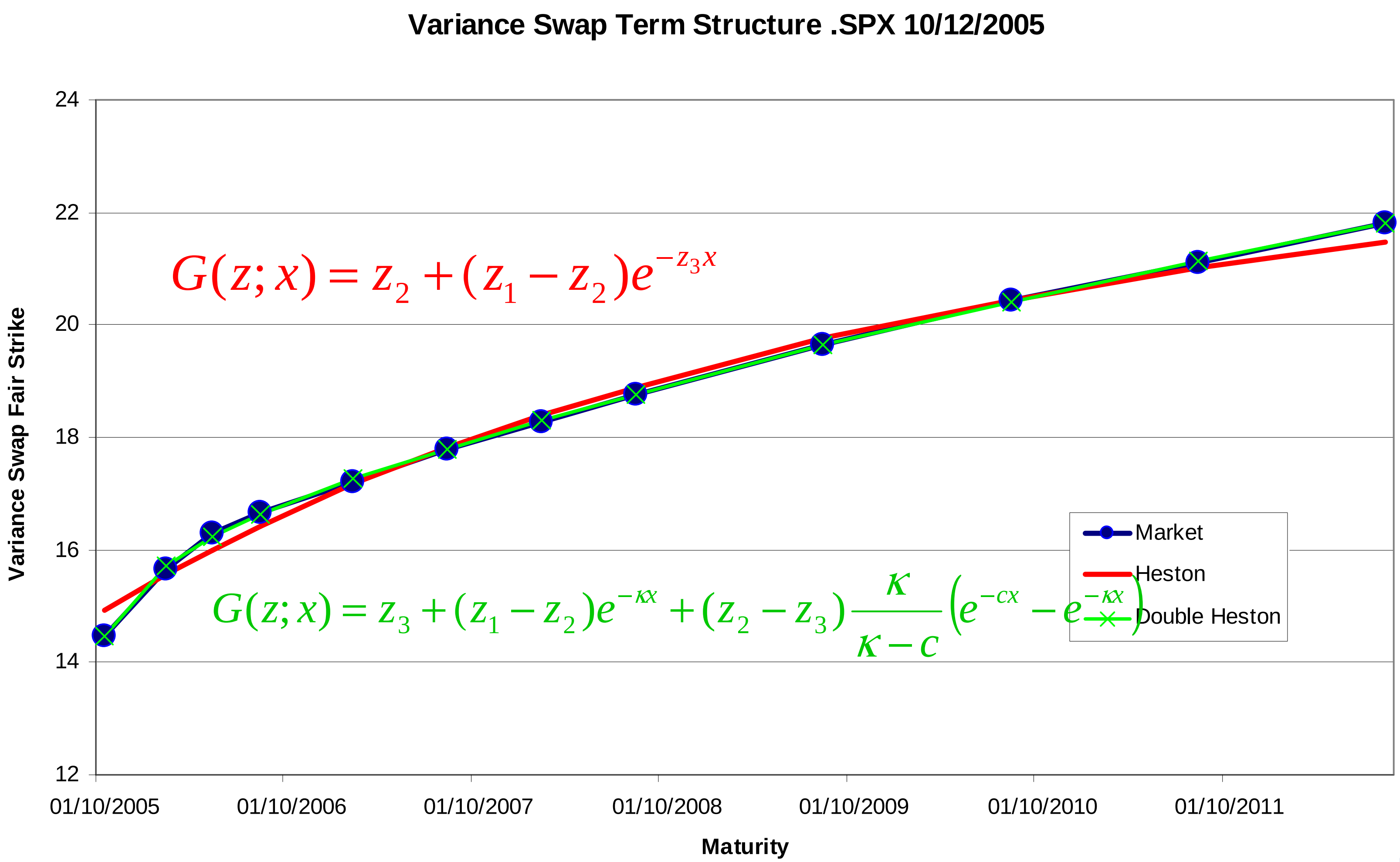
$$\begin{aligned} dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \sigma_1(Z_t)dW_t \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \sigma_2(Z_t)dW_t \\ dZ_t^3 &= \sigma_3(Z_t)dW_t \end{aligned}$$

  which we call “double mean-reverting”.

  – Quite a good fit for most indices (at least up to Mid-2007)

Variance Curve Models

Consistency – Example linear mean-reversion



Variance Curve Models

Consistency – Example linear mean-reversion

- Such a 4F (3 SV + stock) model is discussed in “Volatility Markets” (2006; 2009)
- Intuitively, a more-factor model is only necessary if we want to price options on variance swaps etc.

Variance Curve Models

Term-structure approach

- The next logical step is to model the entire curve u as a process with values in a Hilbert space H.
  - We follow the path laid by Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004) and Teichmann (2005).

Variance Curve Models in Hilbert Spaces

Classic Approach

Variance Curve Models

Term-structure approach

- Let H be a Hilbert space and assume that the variance curve u is given as a mild solution in H up to $\eta>0$ of

$$du_t = \partial_x u_t dt + \sum_{j=1,\ldots,d} b^j(u_t) dW_t^j$$

where the coefficients b are locally Lipschitz vector fields and u is smooth. We denote by the explosion time.

- A *finite dimensional representation* (FDR) of u is a smooth function G and a finite-dimensional diffusion Z such that locally

$$u_t(\cdot) = G(Z_t; \cdot)$$

as elements in H.

Variance Curve Models

Term-structure approach

- The main difference between variance curves and forward curves is that the curves u must remain non-negative (but *can* become zero).
  - The problem is that the “non-negative cone” is a very small set. Indeed it has no interior points.
  - However, if $G(D)$ is a (positive) sub-manifold of H with boundary, then it is sufficient to check whether u stays in $G(D)$. In this case we say $G(D)$ is *locally invariant* for u.
  - If G is moreover invertible, we can also directly construct a (locally) consistent factor model $C=(\mu,\sigma)$ for G.

→ Use Filipovic/Teichmann 2004

Variance Curve Models

Term-structure approach

- The Stratonovic-drift for u is as usual

Frechet-Derivative

$$\beta^0(u) := \partial_x u - \sum_{j=1,\dots,d} D\beta^j(u) \cdot \beta^j(u)$$

such that

$$du_t = \hat{\beta}^0(u_t)dt + \sum_{j=1,\dots,d} \hat{\beta}^j(u_t) \circ dW_t^j$$

Stratonovic-Integral

Variance Curve Models

Term-structure approach

- Theorem (Filipovic/Teichmann 2004)
  The sub-manifold $G(D)$ is locally invariant for u iff

1. We have $G(D) \subset \mathrm{dom}(\partial x)$,
2. In the interior of $G(D)$, we have

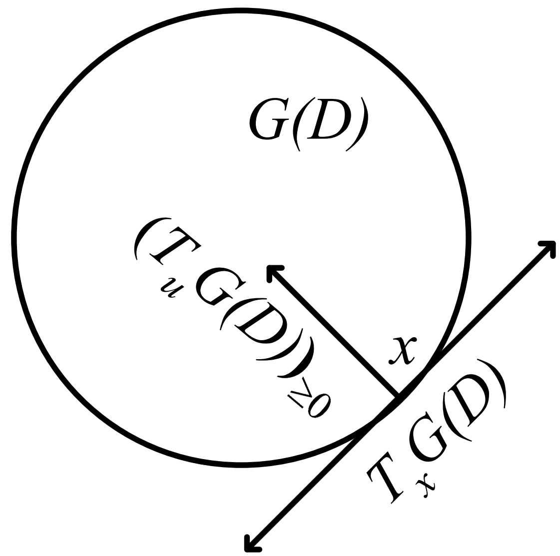
$$\hat{\beta}^j(u) \in T_u G(D) \qquad j = 0, \ldots, d$$

3. On the boundary $\partial G(D)$,

$$\hat{\beta}^0(u) \in (T_u G(D))_{\geq 0}$$

$$\hat{\beta}^j(u) \in T_u \partial G(D) \qquad j = 1, \ldots, d$$

holds.



Variance Curve Models

Term-structure approach

- If we can invert G, then $C=(\mu,\sigma)$ with

$$\sigma^j(z) := \partial_z G^{-1}(\hat{\beta}^j(G(z))$$

$$\mu(z) := \partial_z G^{-1}(\hat{\beta}^0(G)(z(z)) + \sum_{j=1,..d} (\partial_z \sigma^j)(z)\sigma^j(z)$$

  is a locally consistent factor model for G, i.e.

$$u_t(x) = G(Z_t; x)$$

  for

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\dots,d} \sigma^j(Z_t)dW_t^j$$

Market Completeness in Factor Models

"Delta Hedging Works"

Market Completeness in Factor Models

Basics

- Assume a market with state process $X=(S^1,\ldots,S^K;A^1,\ldots,A^M)$ given by the SDE

$$dS_t = \sum_{j=1}^{d} \Sigma_t(S_t, A_t)dW_t^i$$

$$dA_t = \mu_t(S_t, A_t)dt$$

  - S are the tradables in this market
  - A are auxiliary state variables with finite variation (e.g. quadratic variation)
  - This represents a collection of instruments (typically $K>>d$) and some auxiliary variables such that the market is Markov.

When is this market complete?

Market Completeness in Factor Models

Basics

- Classic Definition of market completeness:
  Ability to replicate all *payoffs* (L^1 random variables $H \geq 0$) which are measurable with respect to a reference filtration on which W is extremal.
- Idea:
  - Define the martingale H as $H_t := \mathrm{E}_t[\, H \,]$.
  - Invoke PRP of W to obtain η such that

$$dH_t = \sum_{j=1}^{d} \eta_t^j \, dW_t^j$$

  - If $\Sigma(S_t, A_t)^{-1}$ is S-integrable, then $dW_t = \Sigma(S_t, A_t)^{-1} \, dS_t$ exists, and we obtain

$$dH_t = \sum_{j=1}^{d} \eta_t^j \left(\Sigma(S_t, A_t)^{-1} dS_t \right)^j =: \sum_{i=1}^{K} h_t^i \, dS_t^i$$

Market Completeness in Factor Models

Basics

- The issue is that this requires that Σ has full rank.
  - This is often violated, in particular in Heston-type models.
  - Allows only one hedging instrument for 1F diffusions

- The scope of this approach is also too wide:
  - We are usually not interested in general W-measurable payoffs: we want to replicate payoffs which are functional forms of the path of market observables
    → measurable w.r.t. the filtration generated by $X=(S,A)$.

Market Completeness in Factor Models

Relevant Payoffs

- <u>Definition:</u>
  The market of *relevant payoffs* is the market of all L^1 random variables $H \geq 0$ which are measurable wrt to Y.
  - We call the smallest T such that $H \in \mathcal{F}_T(S,A)$ the *maturity* of H.
  - We also define $H_t := \mathrm{E}_t[\,H\,]$

- We say *the market of relevant payoffs is complete* if for all $\mathcal{F}_T(S,A)$-measurable H we find an locally S-integrable h such that

$$H = \mathrm{E}[H] + \sum_{i=1}^{K} \int_0^T h_t^i \, dS_t^i$$

(each claim H is *replicable* with S).

Market Completeness in Factor Models

Delta Hedging Works

- <u>Definition:</u> if X is a n-dimensional non-explosive diffusion

$$dX_t = \mu(X_t)dt + \sum_{j=1}^{d} \Sigma^j (X_t) dW_t^i$$

such that

$$\mathrm{P}f_t(x) := \mathrm{E}[f(X_T) \mid X_t = x]$$

is $d{<}X{>}\otimes d\mathrm{P}$-almost surely C^1 for all $f \in C_K^{\infty\,(*)}$, then we say X *weakly preserves smoothness*.

(*) smooth with compact support

Market Completeness in Factor Models

Delta Hedging Works

- Theorem: if X is weakly preserves smoothness, then X is *extremal* on its own filtration, i.e. for all integrable non-negative $H \in \mathcal{F}_T(X)$ there exist locally X-integrable processes h such that

$$H = \mathrm{E}[H] + \sum_{i=1}^{N} \int_0^T h_t^i \, dS_t^i$$

with

$$\begin{aligned} S_t^i &= X_t^i - \int_0^t \mu^i(X_t)\,dt \\ &= X_0^i + \sum_{j=1}^{d} \int_0^t \Sigma^{i,j}(X_t)\,dW_t^j \end{aligned}$$

(we say H is *replicable*).

  – Moreover, if $H_t(x) := \mathrm{E}_t[\, H \mid X_t = x \,]$ is differentiable, then the integrands are “Delta”, i.e. $h_t = \partial_X H_t(X_t)$.

Market Completeness in Factor Models

Delta Hedging Works

- Let $(\phi_n)_n$ be a *Dirac Sequence* of non-negative smooth functions with compact support and $\int \phi_n(x)\,dx = 1$.
- Let f be a Lebesgue-measurable function and

$$f_n(y) := \int f(x)\phi_n(y-x)dx$$

  - Then f_n is smooth and $f_n \to f$ in $L^p(\lambda)$ for $1 \leq p < \infty$ (or $1 \leq p \leq \infty$ if f is bounded)
  - The derivatives of f_n converge in the same way to those of f, if they exist.

Market Completeness in Factor Models

Delta Hedging Works

- Step 1: if X is bounded, then each $H(X_T)$ for $H\geq 0$ in C_K^∞ is replicable.
  - Define

$$H_t(x) := \mathrm{E}[H(X_T) \mid X_t = x]$$

    which is continuous and $d\langle X\rangle \otimes d\mathrm{P}$-almost surely C^1.
  - Assume H_t is C^2, then Ito and the local martingale property yield

$$H(X_T) = \mathrm{E}[H(X_T)] + \sum_{i=1}^{n} \int_0^T \partial_{X^i} H_t(X_t)\, dS_t^i$$

    for the bounded local martingales S.

Market Completeness in Factor Models

Delta Hedging Works

- Define the smooth bounded functions

$$H_t^n(x) := \int H_t(x')\phi_n(x-x')dx'$$

for which

$$dH_t^n(X_t) \quad = \left\{ \partial_t H_t^n + \partial_a H_t^n \mu + \frac{1}{2}\partial_{SS}^2 H_t^n \sigma^T \sigma \right\}(X_t)dt + \partial_x H_t^n(X_t)dS_t$$

- Since $H_n \to H$ in L^2, the left hand side converges in L^2 to a martingale.
- This is implies convergence $L^2(d\langle X\rangle \otimes d\mathrm{P})$ – Null-sets can be ignored.
- To show that indeed L^2,

$$\int_0^T \partial_S H_t^n dS_t \to \int_0^T \partial_S H_t dS_t$$

we find a localizing sequence which bounds X, i.e. the derivatives on the left. Then apply properties of Dirac-Sequence.

Market Completeness in Factor Models

Delta Hedging Works

- Step 2: if X is bounded then any $H(X_T)$ in L^2 is replicable.
  - Use a Dirac Sequence to obtain smooth H_n with compact support each of which we can replicate.

$$H_n = \mathrm{E}[H_n] + \sum_{i=1}^{K} \int_0^T h_t^{i,n} \, dS_t^i$$

  - The left hand side converges in L^2, hence the right hand integrands h^n converge in $d\langle X \rangle \otimes d\mathrm{P}$.

- Step 3: if X is bounded, then any $H(X_{T1};\ldots;X_{Tn})$ in L^2 is replicable.
  - Conditional expectations

Market Completeness in Factor Models

Delta Hedging Works

- Step 4: if X is bounded, then any H in L^2 is replicable.
  - Take a countable representation $\{t_k\}_k$ of Q (the rational numbers).
  - For each set $\{t_1,\ldots t_m\}$, define the L^2 payoffs

$$H^m = \mathrm{E}\left[H \mid X_{t_1},\ldots,X_{t_m}\right]$$

  - Apply Step 3 to get a representation for each H^m.
  - Take the limit in L^2 (tower law) to yield a representation for H.

- Step 5: if X is a local martigale then any H in L^1 is replicable
  - Localize X and H jointly (extremality is a local property).

Market Completeness in Factor Models

Delta Hedging Works

- Question:
  When does a non-explosive diffusion (say, with (μ,Σ) locally Lipschitz) weakly preserve smoothness?

- Lemma
  Assume that X is a unique, strong, non-explosive solution to

$$dX_t = \mu(X_t)dt + \sum_{j=1}^{d} \Sigma^j (X_t) dW_t^j$$

and that (μ,Σ) is C^1 with locally Lipschitz derivatives.
Then, X weakly preserves smoothness.

Market Completeness in Factor Models

Delta Hedging Works

- Proof:
  - Let $X_0=x$. The “derived” processes

$$X_t^k := (\partial_{x^k} X_t^1, \ldots, \partial_{x^k} X_t^n)$$

    exist and are of the form

$$dX_t^k = \sum_{i=1}^n X_t^{r,k} \left\{ \partial_{x^i} \mu(X_t) dt + \sum_{j=1}^d \partial_{x^i} \Sigma^j (X_t) dW_t^j \right\}$$

  - The coefficients are locally Lipschitz, i.e. this equation has a solution.
  - Since X does not explode, neither do the derived processes.
  - For f in C_K^∞ that means that by taking the limit through the expection,

$$\partial_{x^r} \mathrm{E}[f(X_t)] = \mathrm{E}[\partial_{x^r} f(X_t)] = \mathrm{E}[X_t^r f'(X_t)]$$

    is well-defined.

Market Completeness in Factor Models

Delta Hedging Works

- <u>Theorem (All factor models are locally extremal)</u>
  If the mild solution u in H to

  $$du_t = \partial_x u_t dt + \sum_{j=1,\ldots,d} b^j(u_t) dW_t^j$$

  admits a Finite Dimensional Representation (G,Z), with a locally invertible G, then the market of relevant payoffs is *locally extremal*, i.e. there exists a common stopping time $\tau>0$, such that for all L^1 random variables $H\geq 0$ which are measurable wrt to u, there are locally integrable processes h such that

  $$\mathrm{E}_\tau[H] = \mathrm{E}_0[H] + \sum_{i=1}^{n} \int_0^\tau h_t^i (dZ_t^i - \mu^i(Z_t)dt)$$

  – Proof: as seen before, μ and Σ are at least C^2.

Market Completeness in Factor Models

Delta Hedging Works

- Question: can we do better for the case:

$$dX_t = \sum_{j=1}^{d} \Sigma^j (X_t) dW_t^j$$

Or can we find an example of locally Lipschitz S with global solution X which does not preserve smoothness weakly …?

Market Completeness in Factor Models

Delta Hedging Works

- <u>Theorem ("Delta-Hedging works")</u>
  Assume that $X=(S,A)$ is a strong, unique, non-explosive solution to

$$dS_t = \sum_{j=1}^{d} \Sigma_t^j (S_t, A_t) dW_t^j$$
$$dA_t = \mu_t (S_t, A_t) dt$$

  i.e. the vector S represents the tradable instruments in the market.

  – If S weakly preserves smoothness, then the market of all relevant payoffs is complete.

  – If (μ,Σ) is C^1 with local Lipschitz derivatives, then the market of all relevant payoffs is complete.

Market Completeness in Factor Models

Delta Hedging Works

- The actual statement of the previous theorem is:
  - The generic process X is the background diffusion (for example our consistent variance curve model).

$$dX_t = \mu(X_t)dt + \sum_{j=1}^{d} \Sigma^j(X_t)dW_t^i$$

    - Its properties decide whether the basic information structure is extremal.
    - But “replication” can only work if there are tradable instruments.
  - Hence In order to obtain a complete market, we need to be able to express our tradables S as a Markov process

$$dS_t = \sum_{j=1}^{d} \Sigma_t^j(S_t, A_t)dW_t^j$$
$$dA_t = \mu_t(S_t, A_t)dt$$

Market Completeness in Factor Models

Consistent Variance Curve Models

- We are back in our initial classical setting, i.e. we have decided to use a consistent variance curve model such that

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\dots,d} \sigma^j(Z_t)dW_t^j$$

$$u_t(x) = G(Z_t; x)$$

- Assume that Z preserves smoothness weakly.

Market Completeness in Factor Models

Consistent Variance Curve Models

- Define the *variance swap price* function

$$\overline{G}(z;x) := \int_0^x G(z;y)dy$$

and assume that

$$\overline{G}_{t_1,\ldots,t_m}(z) := (\overline{G}(z;t_1),\ldots,\overline{G}(z;t_m))$$

is invertible.

Market Completeness in Factor Models

Consistent Variance Curve Models

- Define $K \geq m$ tradables

$$S^1 := V(T_1), \ldots, S^K := V(T_K)$$

and the finite variation auxiliary process

$$A_t = \int_0^t G(Z_u;0)\,du$$

such that

$$Z_t = \overline{G}^{-1}_{T_1-t,\ldots,T_K-t}\left(S_t^1 - A_t, \ldots, S_t^K - A_t\right)$$

Variance swap **Running realized variance**

(if $T_i < t$, then set the respective component to zero).

Market Completeness in Factor Models
Consistent Variance Curve Models

- For example, to replicate an option on variance

$$H_t = C_t(Z_t, A_t) := \mathrm{E}\left[h\left(\int_0^T \zeta_s ds \right) \Big| \, Z_t ; A_t \right] = \mathrm{E}\left[h\left(\int_t^T \zeta_s ds + A_t \right) \Big| \, Z_t \right]$$

we can write it as

$$C_t(Z_t, V_t(t)) \equiv \widetilde{C}_t(V_t(T_1), \ldots, V_t(T_m), A_t)$$

Since G is C^2 and because Z weakly preserves smoothness, we get

$$d\widetilde{C}_t(\cdots) = \sum_{i=1}^K \partial_{V_i} \widetilde{C}_t(\cdots) dV_t(T_i)$$

Market Completeness in Factor Models

Consistent Variance Curve Models

- This

$$dC_t(\cdots) = \sum_{i=1}^{K} \partial_{V_i} C_t(\cdots) dV_t(T_i)$$

  is the desired hedge in terms of variance swaps.

  – For options on variance, this is a “natural” hedge.

  – It can also be used for standard options (a delta-term for S will appear).

  – For forward started options, correlation (skew) risk should be taken into account.

- In practise, the above “VarSwapDelta” hedging ratios are computed via bumping of the variance swap price.

Market Completeness in Factor Models

Delta Hedging Works

- Theorem (All Variance Curve Factor Models are Locally Complete)
  If the mild solution u in H to

$$du_t = \partial_x u_t dt + \sum_{j=1,\ldots,d} b^j(u_t) dW_t^j$$

  admits a Finite Dimensional Representation (G,Z), with an invertible G and C^2 b, then the market of relevant payoffs is *locally complete*, i.e. there exists a common stopping time $\tau>0$, and a set of maturities $T_1,\ldots,T_K$ such that for all L^1 random variables $H\geq 0$ which are measurable wrt to u, there are locally integrable processes h such that

$$\mathrm{E}_\tau[H] = \mathrm{E}_0[H] + \sum_{i=1}^{K} \int_0^\tau h_t^i dV_t(T_i)$$

  – Proof: invertibility and smoothness of G.

Thank you very much for your attention.

Details on the material presented here can be found in “Volatility Markets: Consistent Modeling, Hedging and Practical Implementation of Variance Swap Market Models”, VDM Verlag (2009).
A forthcoming working paper is planned on market completeness.

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