# Consistent Variance Curve Models
*Theory and Application*

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# Consistent Variance Curve Models
## Outline

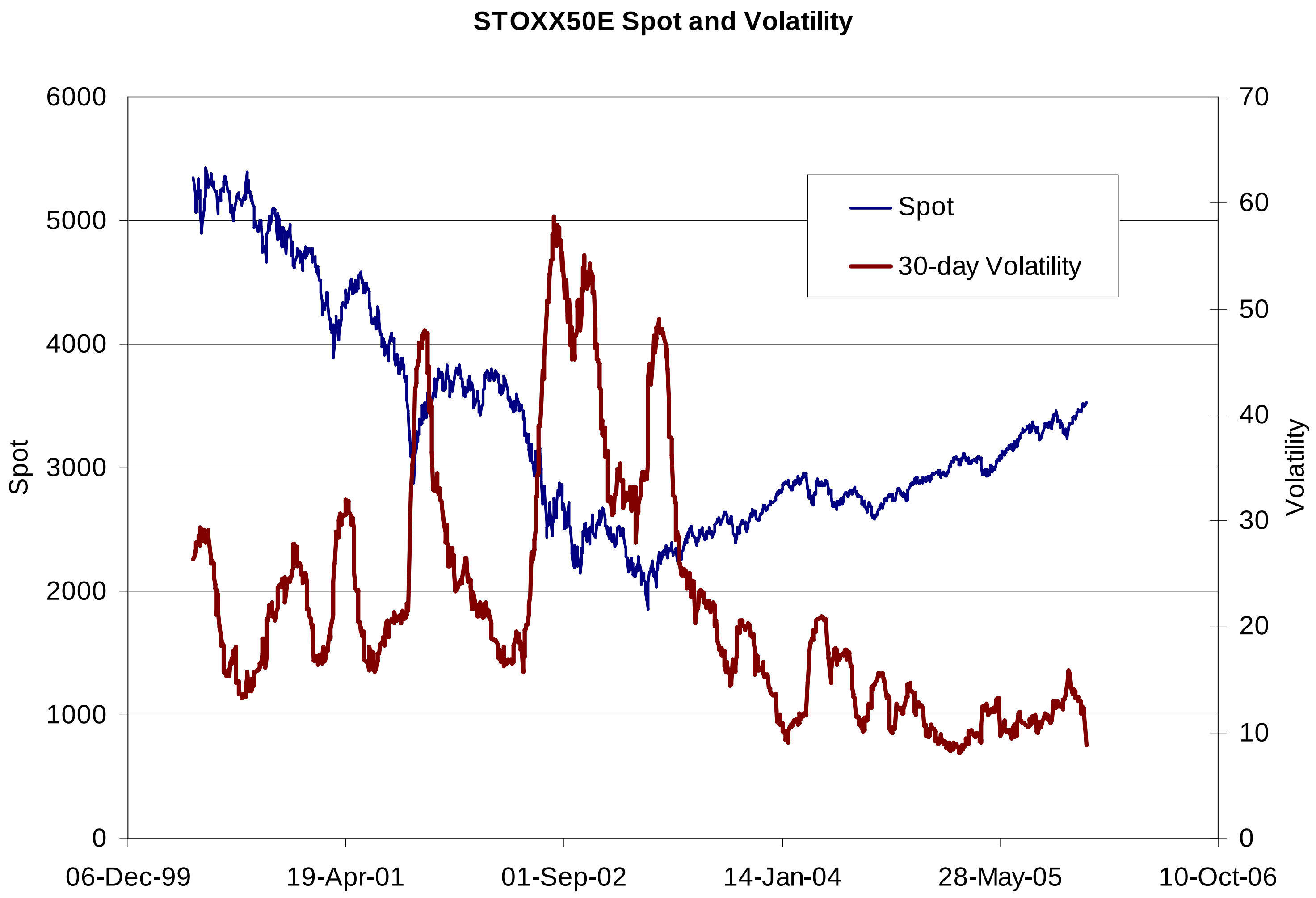
- Introduction to realized variance
- Variance Curve Models
  - General theory
  - Finite-dimensionally parameterized curves
- Examples
- Hedging
- Fitting the market
- Outlook

# Realized Variance

Trading volatility

# Realized Variance

## Introduction

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# Realized Variance
## Introduction

- Equity market investors are interested in “trading volatility”
  - Speculation
  - Hedging
    - Ad-hoc “vega-hedging” against moves in volatility if Black&Scholes-type pricing models are used

- Traditionally, both have been implemented using European options.
- But European options are not very sensitive to volatility once spot moves away from the strike.
  - Why don’t we trade volatility directly?

# Realized Variance
## Introduction

- The *realized variance* of a stock price process $S=(S_t)_t$ over business days $0=t_0<\cdots<t_n=T$ is given as the unbiased estimator

$$\frac{252}{n}\sum_{i=1}^{n}\left(\log\frac{S_{t_t}}{S_{t_{t-1}}}\right)^2$$

  - Inherent "zero-mean" assumption.
  - The $252/n\approx 1/T$ factor "annualizes" the variance.
  - For single stocks, dividends are taken out:

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# Realized Variance
Introduction

- That also makes sense from a “stochastic analysis” viewpoint: If $T$ is fixed but $n\uparrow\infty$, then we see that

$$\langle \log S \rangle_T \approx \sum_{i=1}^{n} \left( \log \frac{S_{t_t}}{S_{t_{t-1}}} \right)^2$$

by definition of the quadratic variation.

  – This is also true if $S$ has a drift and potentially jumps, hence the zero-mean assumption is justified in the limit.

- In the forthcoming discussion, we will assume that realized variance is defined as quadratic varation.

- The error is discussed in Barndorff-Nielsen et al (2004).

# Realized Variance

## Assumptions

- Assume that $S$ is continuous, that it pays no dividends and that the interest rates are zero. Hence, we may write it on a stochastic base $(\Omega, \mathrm{P}, \mathrm{F})$ as

$$
\begin{aligned}
S_t &= \exp\left(X_t - \tfrac{1}{2}\langle X \rangle_t\right) \\
dX_t &= \sqrt{\zeta_t}\, dB_t
\end{aligned}
$$

  - The one-dimensional Brownian motion $B$ is adapted to the filtration $\mathrm{F}$.
  - The *short variance* process $\zeta$ is a predictable, integrable and non-negative.
  - Deterministic rates and proportional dividends can be taken into account (forthcoming “Equity Hybrid Derivatives”, 2006)

- Realized variance is then the non-negative quantity

$$
\langle \log S \rangle_T = \int_0^T \zeta_s \, ds
$$

# Realized Variance
## Variance Swaps

- The simplest product on realized variance is a *variance swap*.
- A variance swap is just a forward on realized variance:
  - At maturity $T$ it pays the realized variance occurred during the life of the contract (usually in exchange for a previously agreed fixed strike $K$).
  - The price $V_t(T)$ of a zero strike variance swap is just the expectation of the realized variance under an equivalent martingale measure

$$V_t(T) := \mathrm{E}\left[\int_0^T \zeta_s ds \mid \mathrm{F}_t\right]$$

# Realized Variance

## Variance Swaps

- If European options are traded for all strikes, the price of a variance swap can in theory be computed in terms of European options using Neuberger's (1990) formula,

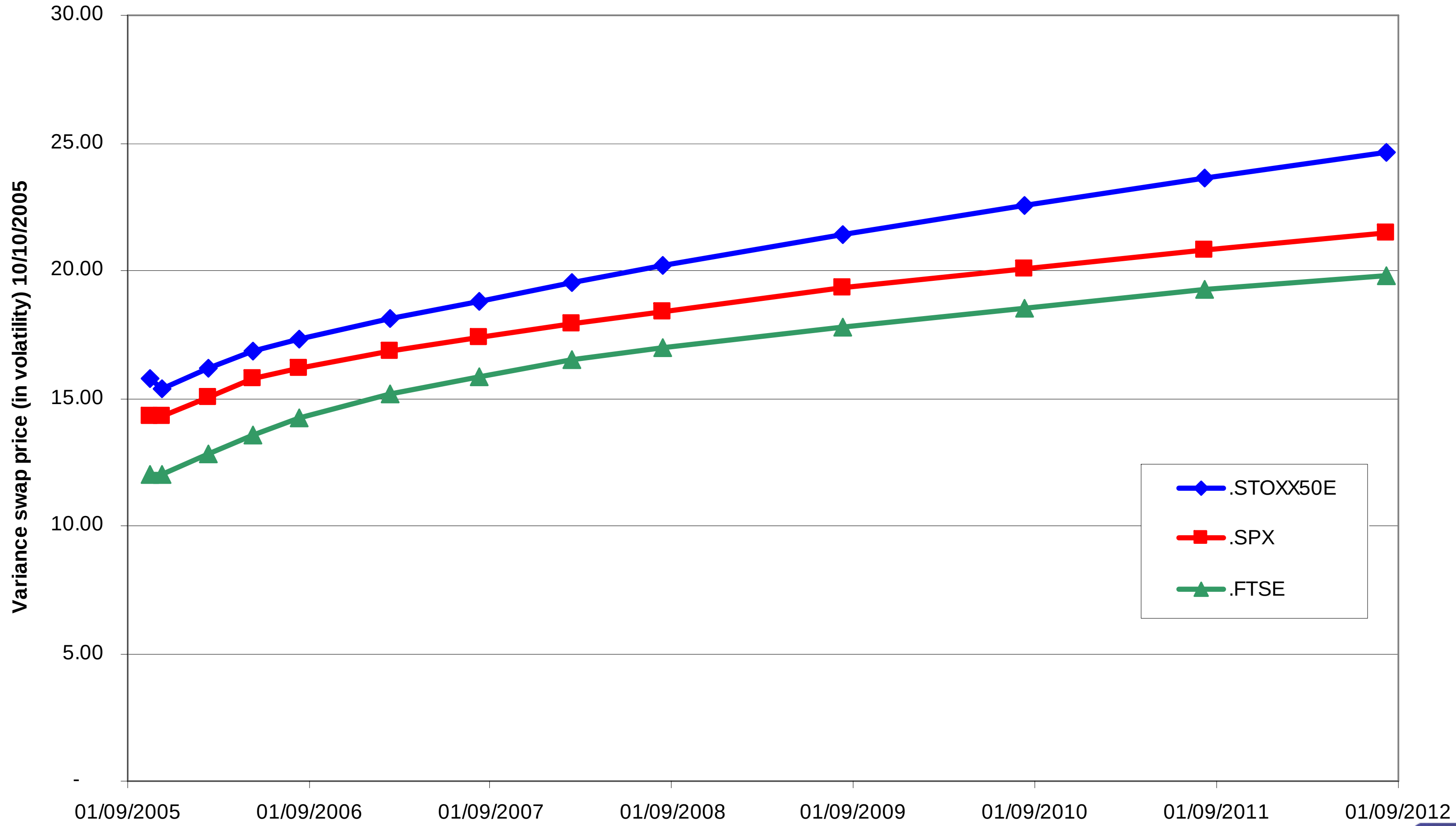
$$\begin{aligned} V_0(T) &= 2\,\mathrm{E}\left[-\int_0^T \sqrt{\zeta_s}\,dB_s + \tfrac{1}{2}\int_0^T \zeta_s\,ds\right] \\ &= 2\,\mathrm{E}\left[S_T - 1 - \log S_T\right] \\ &= 2\left\{\int_0^1 \frac{1}{K^2}\mathrm{Put}(T,K)\,dK + \int_1^\infty \frac{1}{K^2}\mathrm{Call}(T,K)\,dK\right\} \end{aligned}$$

- The formula probably contributes to the fact that variance swaps are now liquidly traded for all major indices.
  - An excellent reference is Demeterfi et al (1999).

# Realized Variance

## Variance Swaps

Prices are quoted in “volatility” $\sqrt{\frac{1}{T}\int_0^T \zeta_s\,ds}$



# Realized Variance

## Variance Swap Markets

- In particular in the US, the variance swap market is very liquid.
  - Spread in terms of volatility is just 0.4 vol points, compared with 0.2 vol points for ATM European options.
  - Bloomberg started quoting variance swaps Jan 06 – until now OTC.

- VIX in the US
  - Volatility index on SPX realized variance
  - The VIX index states the floating one month variance swap price ("fixed-time-to-maturity")
  - Future not very liquid due to problems in replication (constant roll-over of variance swaps involves rolling over of European options if position is hedged).

# Realized Variance

## Beyond Variance Swaps

- Since variance swaps are liquidly traded, there is no need to price them.
- But what about more complex products:
  - *Calls* on realized variance

$$\left(\int_0^T \zeta_s ds - K^2\right)^+$$

  - *Volatility swaps*

$$\sqrt{\int_0^T \zeta_s ds} - K$$

  - But also *forward started options* on the stock

$$\left(S_{T_2} \big/ S_{T_1} - k\right)^+ = \left(\exp\left(\int_{T_1}^{T_2} \sqrt{\zeta_s}\, dB_s - \tfrac{1}{2}\int_{T_1}^{T_2} \zeta_s ds\right) - k\right)^+$$

# Realized Variance

Beyond Variance Swaps

- Very popular: bets on *volatilty of volatility* (VolOfVol) in the form of straddles on realized variance:

$$\left| \int_0^T \zeta_s ds - K^2 \right|$$

  – The strike $K$ is the variance swap strike (“ATM straddle”)

  – Initial “delta” is zero, but high gamma.

- Also sought after: capped calls

$$\min\left\{ (20\% + K)^2, \left( \int_0^T \zeta_s ds - K^2 \right)^+ \right\}$$

# Realized Variance

## Modelling volatility

- How can we develop a model to price & hedge such payoffs?
- The "perfect" model for *all* single-underlying equity products would be a *stochastic implied volatility model*, where the evolution of the implied volatility surface is directly described by a low-factor SDE (Brace et al 2001, Cont et al 2002)
  - Principle idea is (in one parametrization or the other) to write

$$d\sigma_t^{T,K} = A_t(\sigma_t^{T,K};T,K)dt + \sum_{j=1,\dots,d} B_t^j(\sigma_t^{T,K};T,K)dW_t^j$$

  - If such a model is given, the all European option prices at all times are known.
  - Hence, all variance swap prices are known.
  - The stock price is the value of the just maturing zero-strike call.

# Realized Variance
## Modelling volatility

- Unfortunately, all known applications suffer from severe problems.
  1. It is surprisingly complicated to “design” an implied volatilty parametrization which is truly free of arbitrage (no negative butterflies, no negative calendar spreads and boundary conditions).
     - This is essential to guarantee absence of “static” arbitrage.
  2. Even given such a surface, its dynamics cannot be freely chosen: to ensure that the prices of European options are at least local martingales, we have to impose quite complicated constraints on the drift and volatility coefficients of the SDE for the implied volatility.
     - This is necessary to guarantee absence of “dynamic” arbitrage”.
- Other approaches in a similar vein are to model the implied local volatility surface of the stock (Alexander et al 2004), its implied density or simply the European option price surface directly.
  $\rightarrow$ Similar problems arise

# Realized Variance

## Modelling volatility

- However, in case of “option on variance”, intuitively it should be sufficient to model only the variance swaps: the idea is that variance swaps can be used to “delta-hedge” more complex options on realized variance.
  - Of course, to obtain a useful model, we will also want to model the stock price itself and to develop a good concept of “skew”.

- Mathematically, the term-structure of variance swaps reminds on the term-structure of discount bounds in interest rate models.
  - It is therefore tempting to apply concepts from interest rate theory to the pricing of options on variance.

# Variance Curve Models

Modelling volatility

# Variance Curve Models

## Program

- Instead of starting with $S$ as in classic stochastic volatility models, let us first specify the dynamics of the variance swaps .
- Then, construct a (local) martingale $S$ which has the correct quadratic variation.
- The correlation between the Brownian motion which drives $S$ and the variance curve will function as a skew parameter.
- The model shall also yield hedging ratios in terms of variance swaps; we discuss details on hedging in Markovian models.

# Variance Curve Models

## Forward Variance

- Variance swap prices are increasing with maturity $T$.
  - Their price at a later time $t$ also depends on the past realized variance.
- To alleviate these unpleasant properties, note that

$$V_t(T) = \mathrm{E}\left[\int_0^T \zeta_s ds \mid \mathrm{F}_t\right] = \int_0^T \mathrm{E}[\zeta_s \mid \mathrm{F}_t] ds$$

can be differentiated in $T$ to define the *forward variance*

$$\underbrace{v_t}_{\text{Observation time}}(\underbrace{T}_{\text{Maturity}}) := \partial_T V_t(T) = \mathrm{E}[\zeta_T \mid \mathrm{F}_t]$$

- Note the similarity to the *forward rate* in interest rate theory.
- An important property is that forward variance can be zero.

# Variance Curve Models

Classic approach

- Assume we have a driving $d$-dimensional extremal Brownian motion $W$ on the space $(\Omega,\mathrm{P},\mathrm{F})$.
- Definition
  A family $v=(v(T))_{T\geq 0}$ is called a [local] *Variance Curve Model* if

1. For each $T>0$, the process $v(T)=(v_t(T))_{t\in[0,T]}$ is a non-negative [local] martingale:

$$dv_t(T) = \sum_{j=1,\dots,d} \beta_t^j(T)\,dW_t^j \qquad \beta^j(T) \in L^{\mathrm{loc}}$$

2. For each $T>0$, the initial variance swap prices are finite, i.e.

$$V_0(T) = \int_0^T v_0(s)\,ds < \infty$$

3. The curve $v_t(t)$ is left-continuous.

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# Variance Curve Models

## Classic approach

- Properties
  - The price processes of variance swaps,

$$V_t(T) := \int_0^T v_t(s)ds$$

    are [local] martingales.

  - The *short variance process*

$$\zeta_t := v_t(t)$$

    is well defined, integrable and non-negative.

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# Variance Curve Models

Classic approach

- Properties
  Given any standard Brownian motion $B$ on $(\Omega, \mathrm{P}, \mathrm{F})$, the process

$$dX_t = \sqrt{\zeta_t}\, dB_t$$

  is a square-integrable martingale, so the via $B$ *associated stock price*

$$S_t := \exp\left( X_t - \tfrac{1}{2} \langle X \rangle_t \right)$$

  is a local martingale.

  – $B$ represents the *correlation structure* of $S$ with $v$.

- Theorem
  For each variance curve model $v$ and each Brownian motion $B$, the market

$$\left( S; (V(T))_{T \geq 0} \right)$$

  is free of arbitrage.

# Variance Curve Models

Classic approach – Musiela-Parametrization

- As in interest rates, it is more convenient to work with fixed time-to-maturities $x := T - t$. Hence we define the *Musiela parameterization*

$$u_t(x) := v_t(t+x)$$

- <u>Starting in Musiela-parametrization</u>
  - Assume that $\sum_{j=1,\dots,d} \int_0^\infty \int_t^\infty \partial_T \beta_t(T)^2 \, dT \, dt < \infty$
    Then,

$$du_t(x) := \partial_x u_t(x)dt + \sum_{j=1,\dots,d} b_t^j(x)dW_t^j$$

defines a local variance curve model in Musiela-parametrization.

# Variance Curve Models

## Classic approach – Fitting the market

- If $v$ is represented as an exponential,

$$u_t(x) := \exp(w_t(x))$$

it allows us to fit the model easily to an observed market forward variance curve $m_0$ by setting $w_0 := \log m_0$ (cf. Dupire, 2004).

  – This construction does not allow $u$ to become zero and therefore excludes classic stochastic volatility model such as Heston's.

- Another approach is to use an existing model $u^{\text{base}}$ and set

$$u_t(x) := \frac{m_0(t+x)}{u_0^{\text{base}}(t+x)} u_t^{\text{base}}(x)$$

- In both cases mind effects on the martingale property of $S$.

# Variance Curve Models

## Variance Curve Functionals

- Problems with a specification with general integrands $b(T)$:
  - It is complicated to check whether $u$ remains non-negative.
  - In practice, it is not clear how to handle such integrands computationally.
- Ideally, we want to write

$$u_t(x) := G(Z_t; x)$$

for some suitable non-negative function $G$ and an $m$-dimensional Markov-process $Z$.

- The function is the "interpolation function" for the forward variances.

# Variance Curve Models

## Variance Curve Functionals

- Definition

1. A non-negative $C^{2,2}$-function $G: D x R^{+} \to R^{+}$ is called a *Variance Curve Functional* if

$$\int_0^T G(z;x)dx < \infty$$

for all $T$ and $z \in D$ where $D$ is an open set in $R_{\geq 0}^{m}$.

2. We denote by $\Xi$ the set of all $C=(\mu,\sigma)$ for which the SDE

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\ldots,d} \sigma^j(Z_t)dW_t^j$$

starting at any point $Z_0 \in D$ has a unique solution $Z$ which stays in $D$.

– Time-dependency is included in this setup.

# Variance Curve Models

## Variance Curve Functionals

- Definition
  We call $C=(\mu,\sigma)\in\Xi$ a *consistent factor model* for $G$ if for any $Z_0\in D$,

$$u_t(x) := G(Z_t; x)$$

  defines a local variance curve model.

- Theorem
  This is the case if and only if $Z$ stays in $D$ and if

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^T\sigma(z)\,\partial^2_{zz} G(z;x)$$

  holds.

# Variance Curve Models
## Variance Curve Functionals

- <u>Local Correlation and the Markov property</u>
  Given a consistent factor model $C=(\mu,\sigma)\in\Xi$ and a "correlation function" $\rho:R^{+}xD\rightarrow[-1,1]^{d}$ with $|\rho|=1$, we can always define

$$dS_t = \sum_{j=1,\dots,d} S_t \rho^{j}(S_t;Z_t)\left\{\sqrt{G(Z_t;0)}dW_t^{\,j}\right\}$$

  such that the process $(S,Z)$ is Markov and $S$ is a local martingale (note that the SDE does not explode).
  - The Markov property is essential for market completeness (see Buehler/Teichmann 2006).

- <u>Local Volatility</u>
  Local-Stochastic volatility "mixture models" are also part of this framework: they correspond to the case where $S$ is one of the factors of $Z$.

# Variance Curve Models

Term-structure approach

- The next logical step is to model the entire curve $u$ as a process with values in a Hilbert space $\mathrm{H}$.
  - The follows the path laid by Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004) and Teichmann (2005).

- We omit this discussion here and refer to Buehler (2006).

# Variance Curve Models

## Term-structure approach

- The next logical step is to model the entire curve $u$ as a process with values in a Hilbert space $\mathrm{H}$.
  - We follow the path laid by Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004) and Teichmann (2005).

- The main difference between variance curves and forward curves is that the curves $u$ must remain non-negative (but *can* become zero).
  - The problem is that the “non-negative cone” is a very small set. Indeed it has no interior points.
  - However, if $G(D)$ is a sub-manifold with boundary of $\mathrm{H}$, then it is sufficient to check whether $u$ stays in $G(D)$. In this case we say $G(D)$ is *locally invariant* for $u$.
  - If $G$ is moreover invertible, we can also directly construct a (locally) consistent factor model $C=(\mu,\sigma)$ for $G$.

# Variance Curve Models

## Term-structure approach

- Assume that the variance curve $u$ is given as a solution in $\mathrm{H}$ to

$$du_t = \partial_x u_t dt + \sum_{j=1,\dots,d} b^j(u_t) dW_t^j$$

where the coefficients $\beta$ are locally Lipschitz vector fields.

- The Stratonovic-drift for $u$ is as usual

Frechet-Derivative

$$\beta^0(u) := \partial_x u - \sum_{j=1,\dots,d} D\beta^j(u) \cdot \beta^j(u)$$

such that

$$du_t = \hat{\beta}^0(u_t)dt + \sum_{j=1,\dots,d} \hat{\beta}^j(u_t) \circ dW_t^j$$

Stratonovic-Integral

# Variance Curve Models

## Term-structure approach

- Theorem (Filipovic/Teichmann 2004)
  The sub-manifold $G(D)$ is locally invariant for $u$ iff

1. We have $G(D) \subset \mathrm{dom}(\partial x)$,
2. In the interior of $G(D)$, we have

$$\hat{\beta}^j(u) \in T_u G(D) \qquad j = 0, \ldots, d$$

3. On the boundary $\partial G(D)$,

$$\hat{\beta}^0(u) \in (T_u G(D))_{\geq 0}$$

$$\hat{\beta}^j(u) \in T_u \partial G(D) \qquad j = 1, \ldots, d$$

holds.

# Variance Curve Models

## Term-structure approach

- If we can invert $G$, then $C=(\mu,\sigma)$ with

$$\sigma^j(z) := \partial_z G^{-1}(\hat{\beta}^j(G(z))$$

$$\mu(z) := \partial_z G^{-1}(\hat{\beta}^0(G(z)) + \sum_{j=1,..d} (\partial_z \sigma^j)(z)\sigma^j(z)$$

is a consistent factor model for $G$.

# Applications

How to model variance curves

# Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- <u>Example</u>
  A very basic example is the “linearly mean-reverting” functional:

„Speed of mean-reversion“

$$G(z;x) = z_2 + (z_1 - z_2)e^{-z_3 x}$$

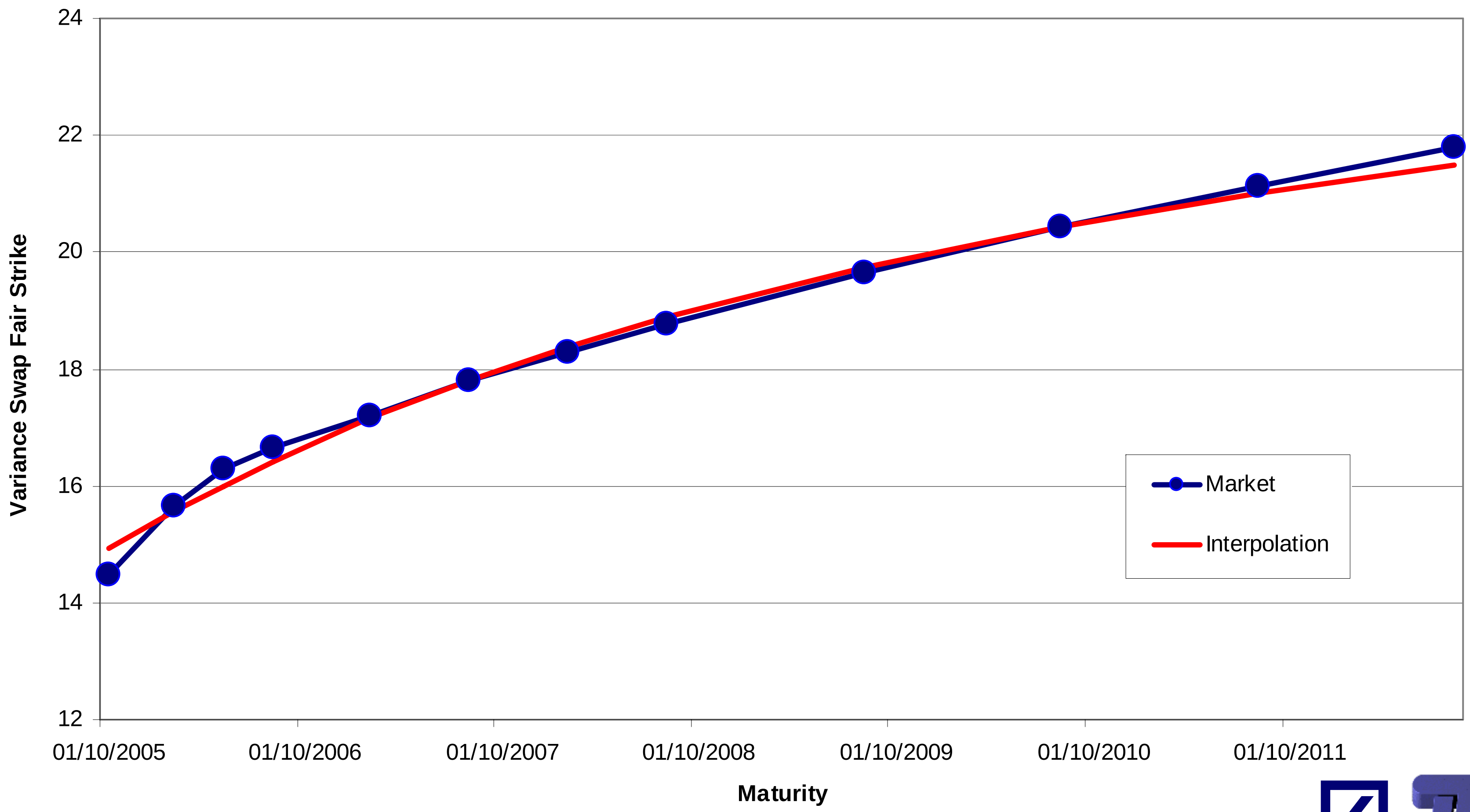
„Long variance“ „Short variance“

for $z_1 \geq 0$ and $z_2, z_3 > 0$.

# Variance Curve Models

## Variance Curve Functionals – Linear mean-reversion

**Variance Swap Term Structure .SPX 10/12/2005**

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# Variance Curve Models

## Variance Curve Functionals – Linear mean-reversion

- Question: What dynamics can a consistent process $Z=(Z_1, Z_2, Z_3)$ have?
- The coefficients $\mu$ and $\sigma$ have to satisfy

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^T\sigma(z)\partial^2_{zz}G(z;x)$$

1. First, we see that
$$\partial^2_{z_3 z_3} G(z;x) = (z_1 - z_2)x^2 e^{-z_3 x}$$
Since no term $x^2e^x$ appears on the left hand side, we must have $\sigma_3=0$.
2. The same line of thought applied to
$$\partial_{z_3} G(z,x) = -(z_1 - z_2)xe^{-z_3 x}$$
shows that we also have $\mu_3=0$.

**Hence, the speed of mean-reversion cannot be stochastic.**

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# Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- For the other two parameters, we find that while $\sigma$ is unconstrained,

$$\mu_2(z) = 0$$
$$\mu_1(z) = z_3(z_2 - z_1)$$

In other words: the only consistent processes for this choice of $G$ are of Heston-type

**Linear mean-reversion drift**

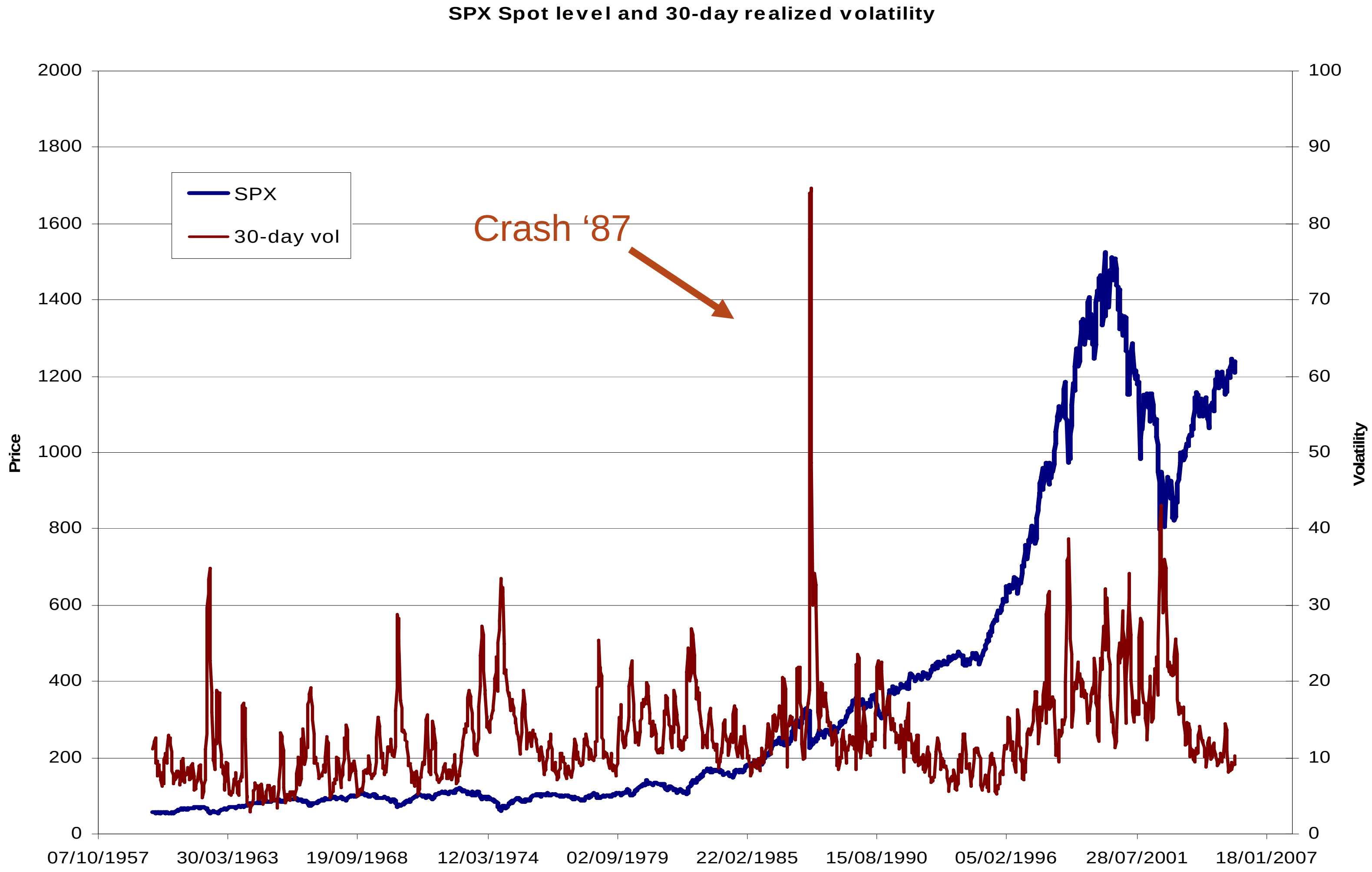
$$\begin{aligned} d\zeta_t &= \kappa(\theta_t - \zeta_t)dt + \sigma_1(\zeta_t, \theta_t)dW_t \\ d\theta_t &= \sigma_2(\zeta_t, \theta_t)dW_t \end{aligned}$$

**VolOfVol can freely be chosen as long as $\zeta$ remains non-negative.**

**Mean-reversion level $\theta$ is a positive martingale.**

# Variance Curve Models

## Why mean-reversion?

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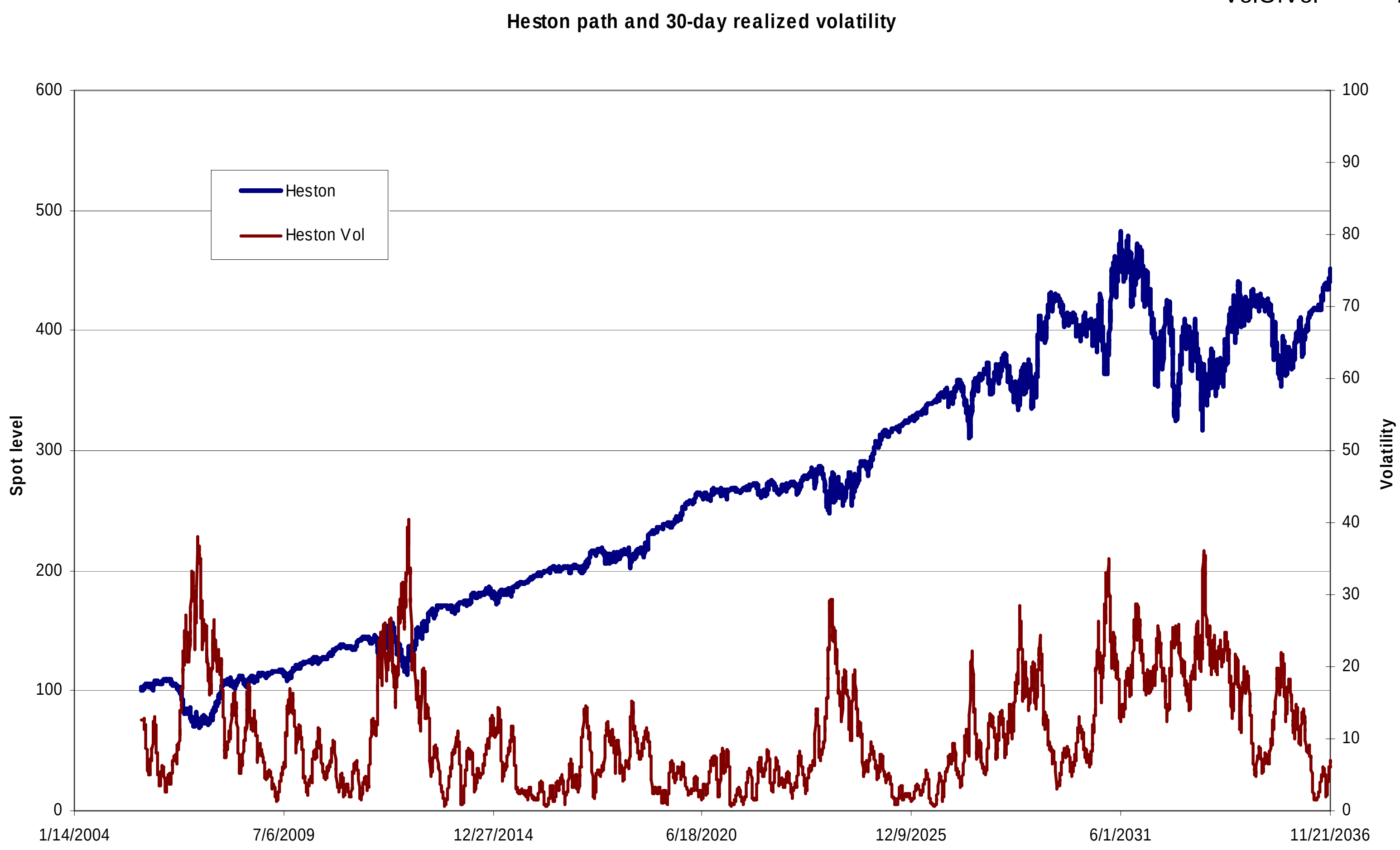
# Variance Curve Models

## Why mean-reversion?

**Unconstrained Calibration**

| | |
|---|---|
| ShortVol | 14.4% |
| LongVol | 28.7% |
| RevSpeed | 0.23 |
| Correlation | -0.74 |
| VolOfVol | 26.3% |

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# Variance Curve Models

## Variance Curve Functionals

- Proposition
  The observation that mean-reversion speeds must be constant holds for all polynomial-exponential functionals, i.e. if $(p_i)_i$ are polynomials

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\sum_{i=1}^{n}p_i(z;x)e^{-z_i x}$$

  then the first $n$ components must be constant (cf. Filipovic 2001 for interest rates).

- A similarly restrictive result can be shown for functionals of the form

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\exp\left\{\sum_{i=1}^{n}p_i(z;x)e^{-z_i x}\right\}$$

  – The parameters in the exponent come in pairs, where one is twice as large as the other (again Filipovic 2001).

# Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

- Another example of the polynomial-exponential class is

$$G(z;x) = z_3 + (z_1 - z_2)e^{-\kappa x} + (z_2 - z_3)\frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right)$$

  – A consistent factor model for this $G$ must have the form

$$\begin{aligned} dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \sigma_1(Z_t)dW_t \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \sigma_2(Z_t)dW_t \\ dZ_t^3 &= \sigma_3(Z_t)dW_t \end{aligned}$$

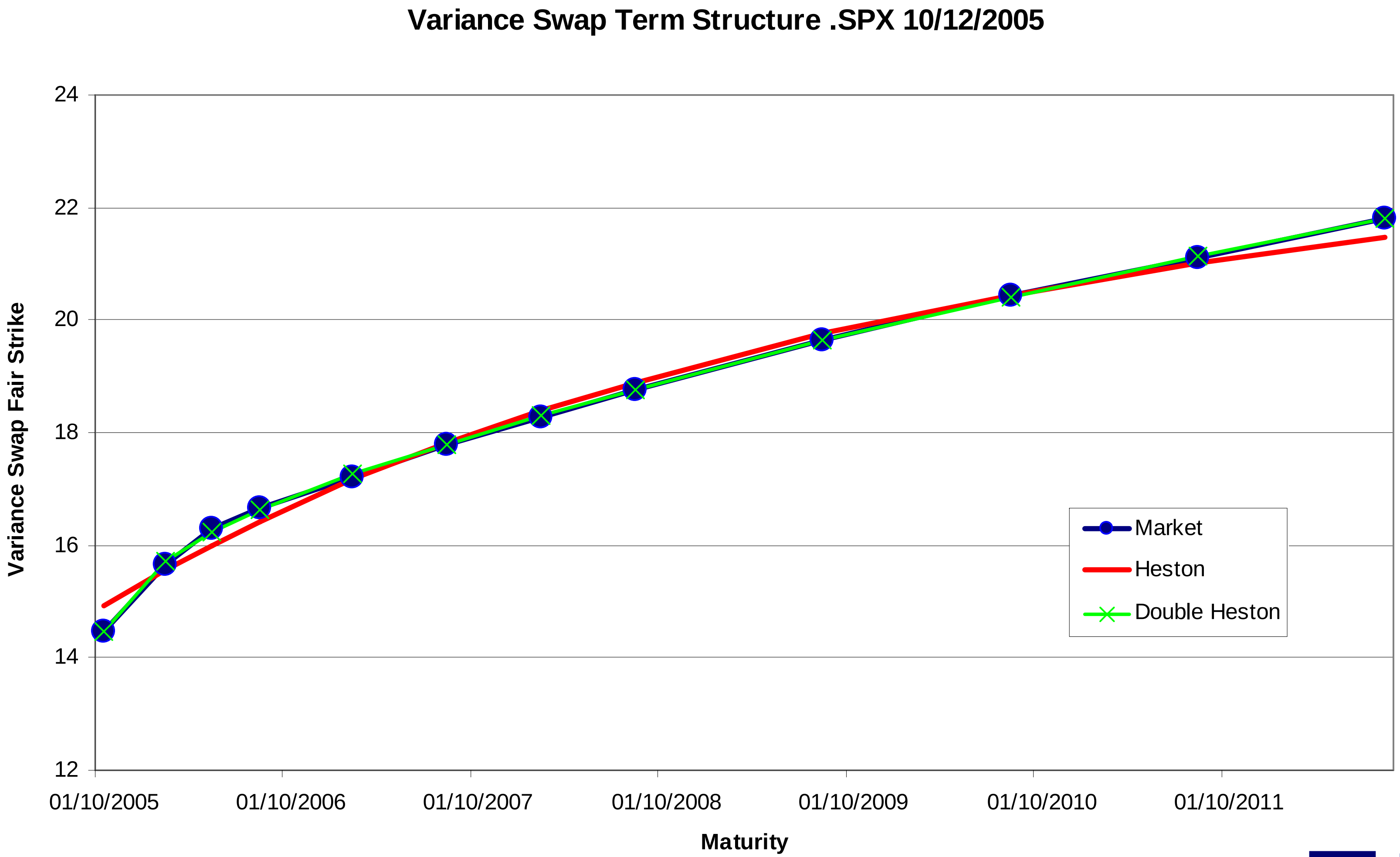
  which we call “double mean-reverting”.

  – Quite a good fit for most indices (at least during the course of the last year).

  – This is in effect Svensson’s interpolation function for interest rates.

# Variance Curve Models

## Variance Curve Functionals – Example linear mean-reversion



# Variance Curve Models

## Variance Curve Functionals – Example linear mean-reversion

- Such a model is discussed in “Equity Hybrid Derivatives” (2006) where we used

$$
\begin{aligned}
dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \nu(Z_t^1)^\alpha d\hat{W}_t^1 \\
dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \mu(Z_t^2)^\beta d\hat{W}_t^2 \\
dZ_t^3 &= \eta Z_t^3 d\hat{W}_t^3
\end{aligned}
$$

  for a correlated vector of Brownian motions.

  – To calibrate it, we first fit the variance curve function itself.

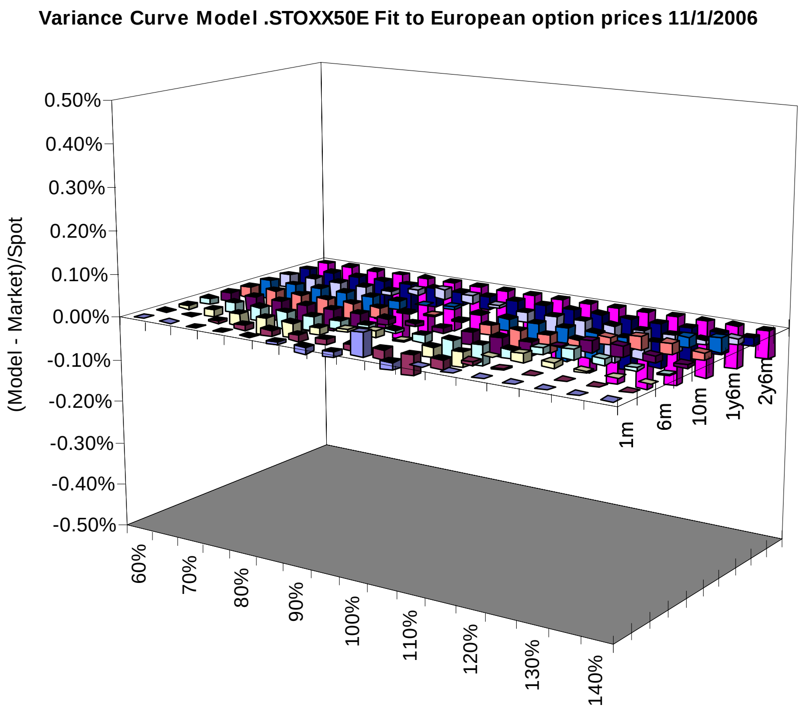
  – In a second step, we use European option prices close to ATM to calibrate the volatility and correlation parameters.

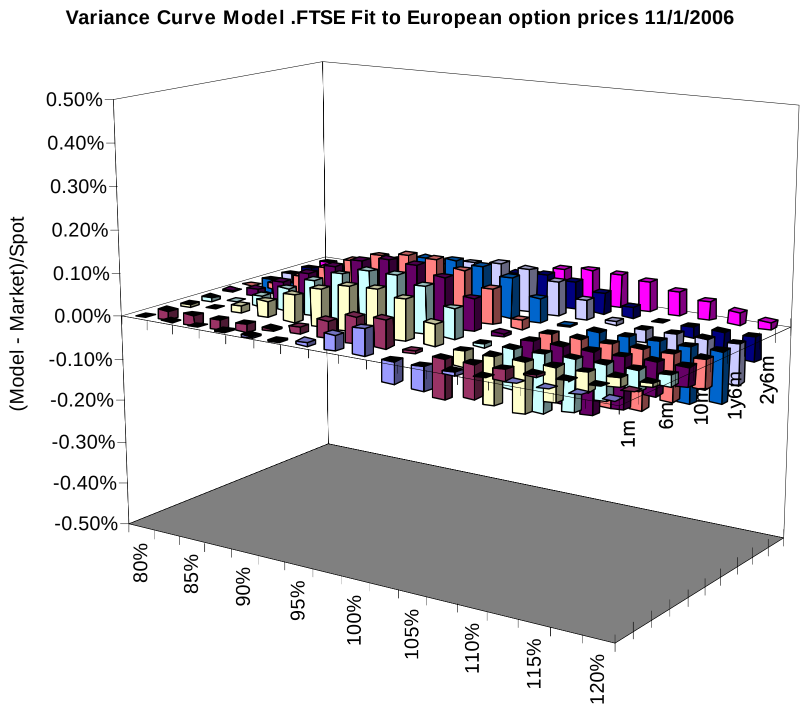
  – Numerically quite tedious.

- Intuitively, a more-factor model is only necessary if we want to price options on variance swaps etc.

# Variance Curve Models

## Variance Curve Functionals – Example linear mean-reversion




Examples of calibrating the three-factor model to real market data.

# Hedging

Using Variance Curve Models to hedge Options on Variance

# Hedging

How to hedge with variance curve models

- Assume we are given a consistent variance curve model,

$$u_t(x) = G(Z_t; x)$$

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\dots,d} \sigma^j(Z_t)dW_t^j$$

$$\zeta_t := u_t(0)$$

- We want to price and hedge an "option on realized variance" with (bounded) European payoff $h$. Its price process is given as

$$\mathrm{E}\left[ h\left( \int_0^T \zeta_s ds \right) \Big| F_t \right]$$

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# Hedging

How to hedge with variance curve models

- Define the *variance swap price* function

$$\overline{G}(z;x) := \int_0^x G(z;y)dy$$

and assume that there exist constant $0<\varepsilon<\tau_1<\cdots<\tau_m$ such that

$$\overline{G}_{t_1,\ldots,t_m}(z) := (\overline{G}(z;t_1),\ldots,\overline{G}(z;t_m))$$

is invertible for all $t_k := \tau_k - \tau$ for $0 \le \tau \le \varepsilon$.

- This then allows to recover $Z$ in any interval $[a,b]$ by

$$Z_t = \overline{G}^{-1}_{T_1-t,\ldots,T_m-t}\left(V_t(T_1)-V_t(t),\ldots,V_t(T_m)-V_t(t)\right)$$

where $T_k := a + \tau_k$.

Variance swap

Running realized variance

# Hedging

How to hedge with variance curve models

- Due to the Markov-property of $Z$, we have

Running realized variance

$$C_t(Z_t, V_t(t)) := \mathrm{E}\left[ h\left( \int_0^T \zeta_s \, ds \right) \Big| \, Z_t; V_t(t) \right]$$

- To hedge this payoff in the interval $[a,b]$, we can write it due to our assumptions on $G$ as

$$C_t(Z_t, V_t(t)) \equiv C_t(V_t(T_1), \ldots, V_t(T_m), V_t(t))$$

such that (under the assumption that $C$ is smooth enough)

$$dC_t(\cdots) = \sum_{k=1}^{m} \partial_{V_k} C_t(\cdots) dV_t(T_k)$$

# Hedging

## How to hedge with variance curve models

- This

$$dC_t(\cdots) = \sum_{k=1}^{m} \partial_{V_k} C_t(\cdots) dV_t(T_k)$$

is the desired hedge of $h$ in terms of variance swaps.

  - For options on variance, this is a “natural” hedge.
  - It can also be used for standard options (a delta-term will appear).
  - For forward started options, correlation (skew) risk should be taken into account.

- In practise, the above “VarSwapDelta” hedging ratios are computed via bumping of the variance swap price.

- Details on hedging in Markovian models in Buehler/Teichmann (2006).

# Using Variance Curves in Practise

Fitting the market

# Variance Curve Models

## Using variance curve models in practise

- To price vanilla options on realized variance, it is sufficient to use a one-factor model which we fit to an observed market curve $m_0$:
- Examples
  - Fitted Log-Normal (for $\kappa=0$ we obtain Dupire 2004)

$$\zeta_t := m_0(t)\frac{e^{\hat{u}_t}}{\mathrm{E}\left[e^{\hat{u}_t}\right]} \qquad d\hat{u}_t = -\kappa\hat{u}_t dt + \sigma dW_t$$

  - Heston:

$$\zeta_t := m_0(t)\frac{\hat{u}_t}{\mathrm{E}\left[\hat{u}_t\right]} \qquad d\hat{u}_t = \kappa(\theta - \hat{u}_t)dt + \sigma\sqrt{\hat{u}_t}\,dW_t$$

- Neither of the above has a closed-form for European options on the stock – calibration of the volatility and correlation parameters more involved.

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# Variance Curve Models

## Using variance curve models in practise

- Fitted Heston model

$$d\zeta_t = \kappa\left(\theta(t) - \zeta_t\right)dt + \nu\sqrt{\zeta_t}\,dW_t$$

  - We set $\theta(t) := \kappa m_0(t) + \partial_t m_0(t)$ which needs to remain non-negative to ensure that the process $\zeta$ is well-defined.
  - Martingale property of $S$ preserved as long as correlation is not positive.

- Another alternative, but not very pretty from a modelling point of view:
  - Use Heston's model for variance and stock and apply deterministic time-change to fit the variance swap market:

$$u_t := \hat{u}_{m_0(t)/\hat{u}_0} \qquad d\hat{u}_t = \left(\hat{u}_0 - \hat{u}_t\right)dt + \nu\sqrt{\hat{u}_t}\,dW_t$$

- European options on the stock price in both cases can be priced reasonably quick using Fourier-Inversion.

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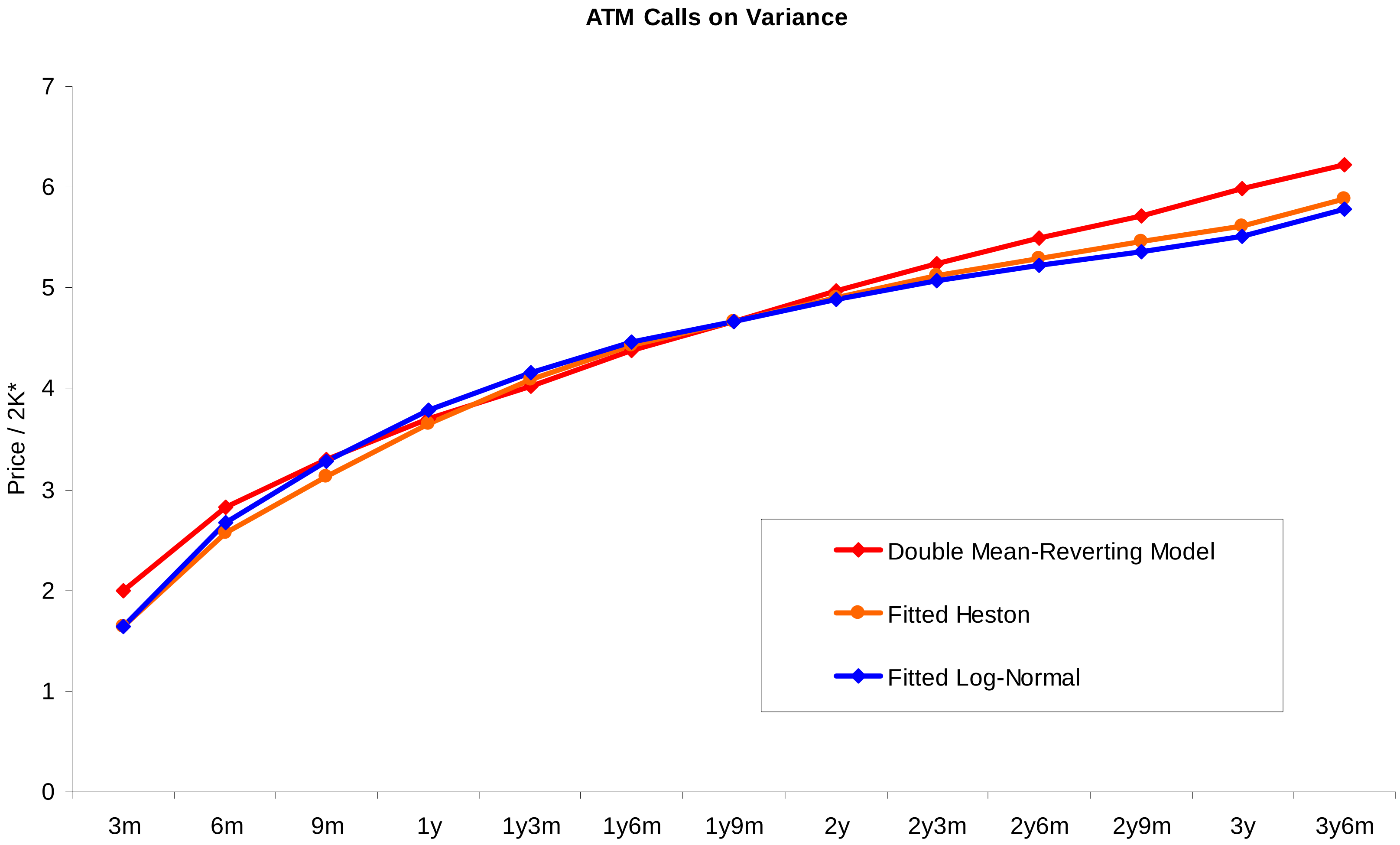
# Variance Curve Models

## Using variance curve models in practise

- To assess the impact of chosing a model, we perform the following ad-hoc test
  - Calibrate the four-factor model presented earlier.
  - Use the variance swap prices of this model as “market”.
  - Match the prices of 1y and 2y ATM calls on variance with the fitted Log-Normal and the fitted Heston model.
- This allows us to compare term-structre and strike dependency between the three models.

# Variance Curve Models

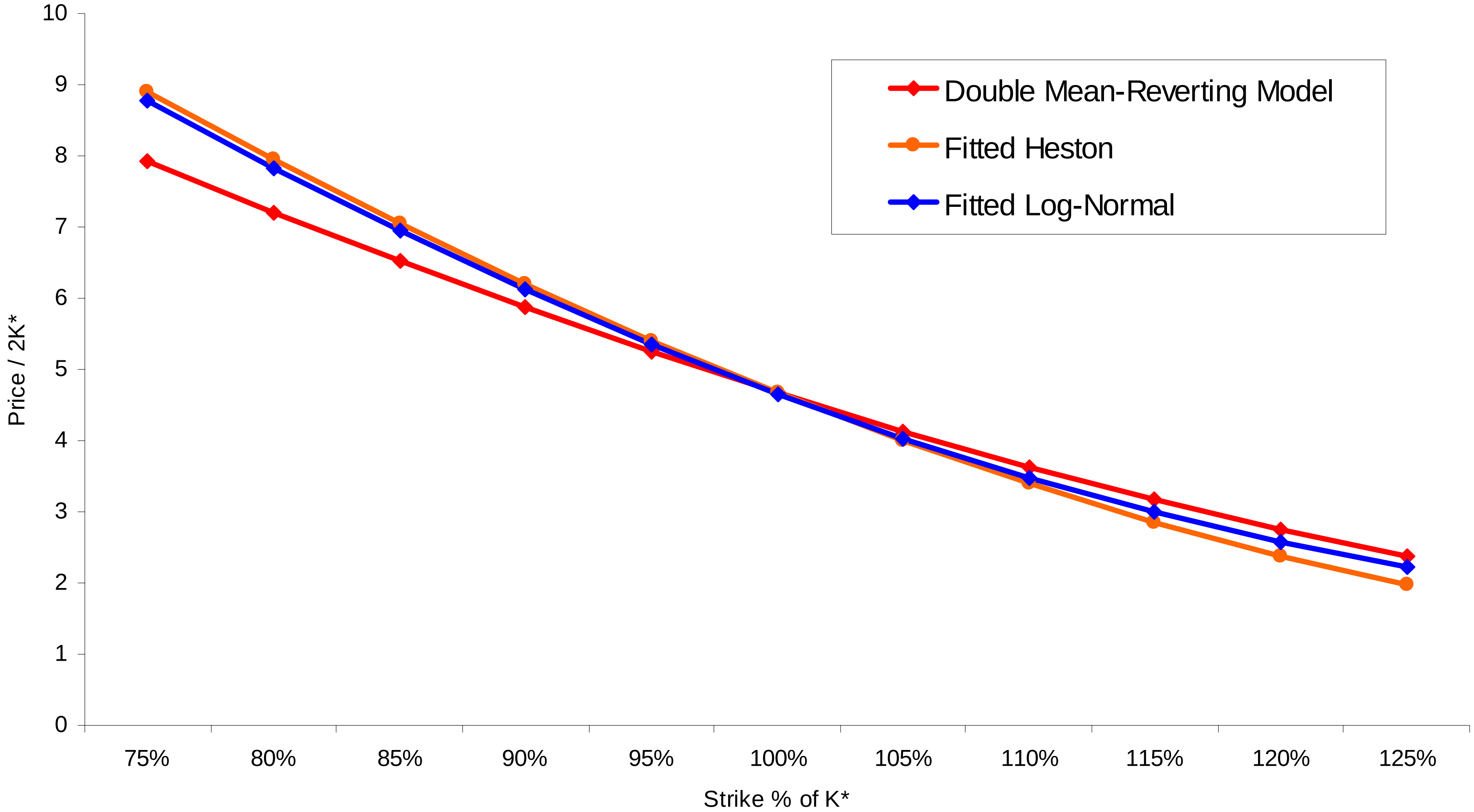
## Using variance curve models in practise



# Variance Curve Models

## Using variance curve models in practise

**1y Calls on Variance for various strikes**



# Variance Curves

Future

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# Variance Curves
## Future

- "Statistical" variance curve models: PCA of historic data
  - Work in progress (Kai Detlefsen, HU Berlin)

- Challenges ahead
  - Is it possible to go from the variance curve model to a stochastic implied volatility model - can the correlation function $\rho$ be deducted from market data?
  - Incorporation of stochastic interest rates and dividends (in particular long-term deals could exhibit strong exposure to stochastic interest rates).
  - Jumps both in the underlying and the variance process (witness S&P return graph earlier).
  - Correlation between the variance curves between different underlyings (our suspicion is that it is actually quite high).

**Thank you very much for your attention.**

Details on the material presented here can be found in “Consistent Variance Curve Models”, Finance and Stochastics (2006), and in the forthcoming “Equity Hybrid Derivatives” (2006).
Hedging in Markovian markets is to be discussed in “Hedging in Factor Models” (joint work with J.Teichmann).

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NB we are generally interested in internship projects !!

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