Modeling Variance Swap Curves: *Theory and Application*

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Consistent Variance Curve Models

Outline

- Introduction to Realized Variance markets
- Variance Curve Models
- Examples and the impact of using multi-factor models
- Fitting the market
- Outlook

Realized Variance

Trading volatility

Realized Variance
Introduction

- Equity market investors are interested in “trading volatility”
  - Speculation
  - Hedging
    - Ad-hoc “vega-hedging” against moves in volatility if Black&Scholes-type pricing models are used

- Traditionally, both have been implemented using European options.
- But European options are not very sensitive to volatility once spot moves away from the strike.
  - Why don’t we trade volatility (or at least variance) directly?

Realized Variance

Introduction

- The *realized variance* of a stock price process $S=(S_t)_t$ over business days $0=t_0<\cdots<t_n=T$ is given as the unbiased estimator

$$\frac{252}{n}\sum_{i=1}^{n}\left(\log\frac{S_{t_t}}{S_{t_{t-1}}}\right)^2$$

  – Inherent “zero-mean” assumption.

  – The $252/n\approx 1/T$ factor “annualizes” the variance.

  – For single stocks, dividends are taken out.

Realized Variance

Introduction

- That also makes sense from a "stochastic analysis" viewpoint:
  If T is fixed but $n \uparrow \infty$, then we see that

$$\langle \log S \rangle_T \approx \sum_{i=1}^{n} \left(\log \frac{S_{t_t}}{S_{t_{t-1}}} \right)^2$$

  by definition of the quadratic variation.
  - This is also true if S has a drift and potentially jumps, hence the zero-mean assumption is justified in the limit.
- In the forthcoming discussion, we will assume that realized variance is defined as quadratic variation.
- The error is discussed in Barndorff-Nielsen et al (2004).

Realized Variance
Assumptions

- Assume that S is strictly positive, continuous, that it pays no dividends and that the interest rates are zero. Hence, we may* write it on a stochastic base $(\Omega, \mathrm{P}, \mathrm{F})$ as

$$\begin{aligned} S_t &= \exp\left(X_t - \tfrac{1}{2}\langle X \rangle_t\right) \\ dX_t &= \sqrt{\zeta_t}\, dB_t \end{aligned}$$

  - The one-dimensional Brownian motion B is adapted to the filtration F.
  - The *short variance* process ζ is a predictable, integrable and non-negative.
  - Deterministic rates and proportional dividends can be taken into account (forthcoming “Equity Hybrid Derivatives”, 2006)

- Realized variance is then the non-negative quantity

$$\langle \log S \rangle_T = \int_0^T \zeta_s \, ds$$

* If additionally its quadratic variation is absolutely continuous w.r.t. λ.

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Realized Variance
Variance Swaps

- The simplest product on realized variance is a *variance swap*.
- A variance swap is just a forward on realized variance:
  - At maturity T it pays the realized variance occurred during the life of the contract (usually in exchange for a previously agreed fixed strike K).
  - The price $V_t(T)$ of a zero strike variance swap is just the expectation of the realized variance under an equivalent martingale measure:

$$V_t(T) := \mathrm{E}\left[\int_0^T \zeta_s ds \mid \mathrm{F}_t \right]$$

  - It can be hedged using a static position in a log-contract and delta-hedging with $1/S_t$:

$$\frac{1}{2}\int_0^T \zeta_t dt = \int_0^T \frac{1}{S_t^2} d\langle S \rangle_t = \int_0^T \frac{1}{S_t} dS_t - \log \frac{S_T}{S_0}$$

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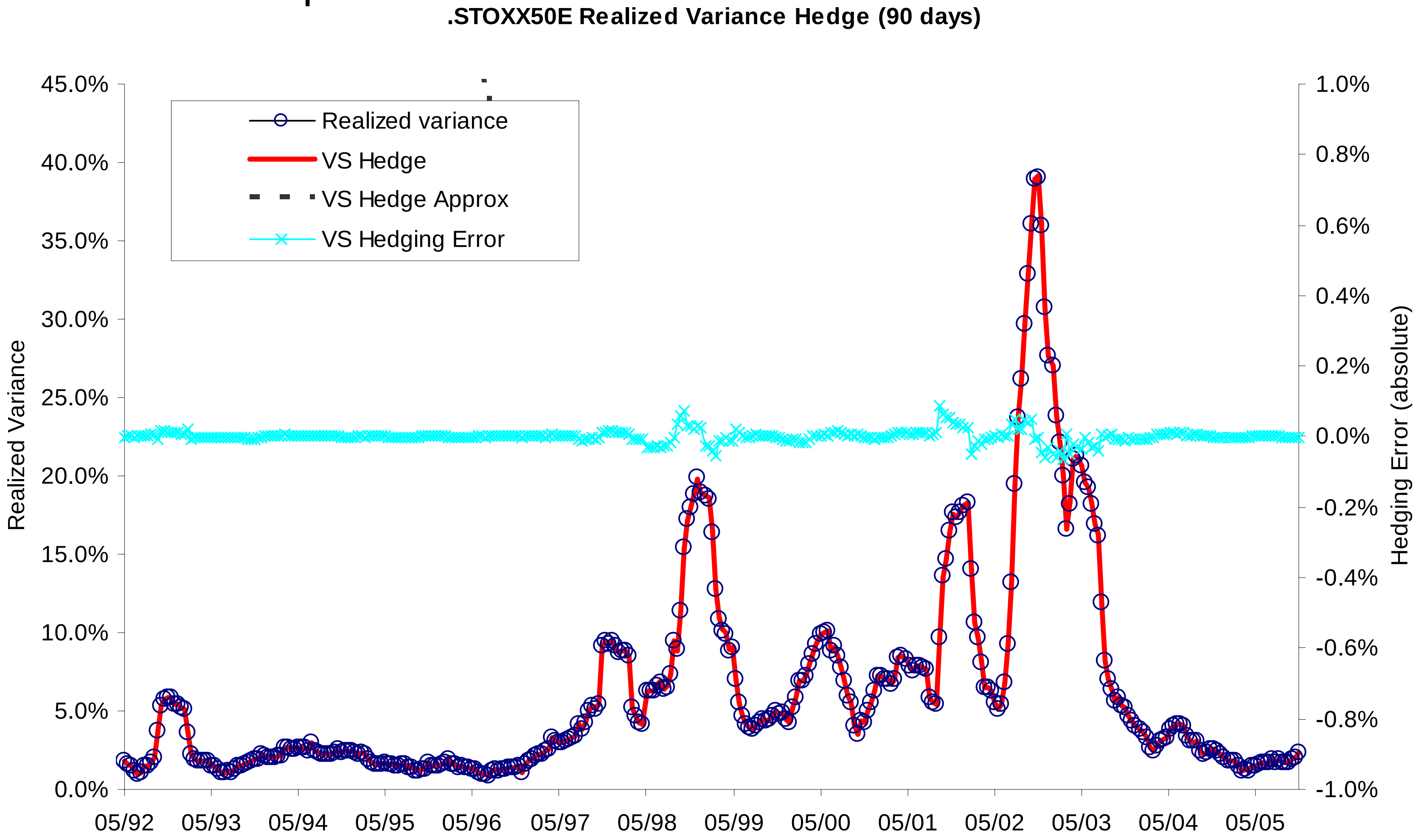
Realized Variance

Variance Swaps



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Realized Variance

Variance Swaps

- The convex European log-payoff is usually approximated by a series of calls and puts,

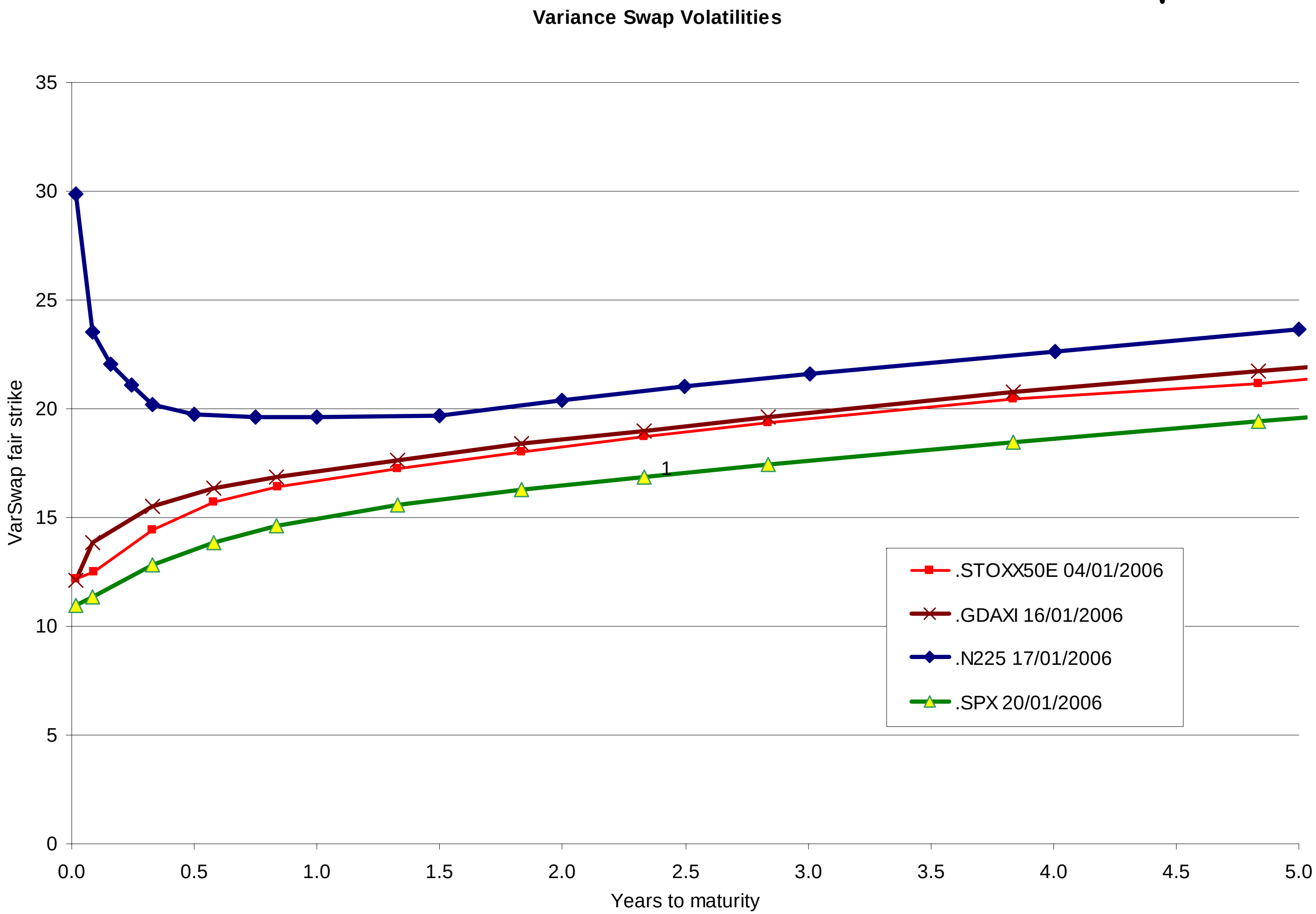
$$
\begin{aligned}
V_0(T) &= 2\,\mathrm{E}\left[-\int_0^T \sqrt{\zeta_s}\,dB_s + \tfrac{1}{2}\int_0^T \zeta_s\,ds\right] \\
&= 2\,\mathrm{E}\left[S_T - 1 - \log S_T\right] \\
&= 2\left\{\int_0^1 \frac{1}{K^2}\mathrm{Put}(T,K)\,dK + \int_1^\infty \frac{1}{K^2}\mathrm{Call}(T,K)\,dK\right\}
\end{aligned}
$$

- The formula probably contributes to the fact that variance swaps are now liquidly traded for all major indices.
  - An excellent reference is Demeterfi et al (1999).

Realized Variance

Variance Swaps

Prices are quoted in "volatility" $\sqrt{\frac{1}{T}\int_0^T \zeta_s ds}$



Realized Variance

Variance Swap Markets

- In particular in the US, the variance swap market is very liquid.
  - Spread in terms of volatility is just 0.4 vol points, compared with 0.2 vol points for ATM European options.
  - STOXX50E around 0.5 versus 0.25 points
  - Bloomberg started quoting variance swaps Jan 06 – until now OTC.

- VIX in the US
  - Volatility index on SPX realized variance
  - The VIX index states the floating one month variance swap price (“fixed-time-to-maturity”)
  - Future not very liquid due to problems in replication (constant roll-over of variance swaps involves rolling over of European options if position is hedged).

Realized Variance

Beyond Variance Swaps

- Since variance swaps are liquidly traded, there is no need to price them.
- Similarly, we do not need particular models to price other recently revived payoffs (they can mostly also be replicated using "model-free" arguments):
  - *Gamma swaps (or weighted variance swaps)*

$$\frac{252}{n}\sum_{i=1}^{n}\frac{S_{t_{t-1}}}{S_0}\left(\log\frac{S_{t_t}}{S_{t_{t-1}}}\right)^2$$

  - *Conditional variance swaps*

$$\frac{1}{n}\sum_{i=1}^{n}1_{L\le S_{t_{t-1}}\le U}\left(252\left(\log\frac{S_{t_t}}{S_{t_{t-1}}}\right)^2-K^2\right)$$

Realized Variance

Beyond Variance Swaps

- But what about more complex products:
  - *European options* on realized variance such as plain calls

$$\left(\frac{1}{T}\int_0^T \zeta_s ds - K^2\right)^+$$

  - *Options on Variance Swaps*

$$\left(V_t(T_R,T) - kV_0(T_R,T)\right)^+ \Big/ T - T_R \qquad V_t(T_R,T) := \mathrm{E}\left[\int_{T_R}^T \zeta_t dt \mid F_{T_R}\right]$$

  - But also *forward started options* on the stock (cf. Bergomi 2005)

$$\left(S_{T_2}\Big/S_{T_1} - k\right)^+ = \left(\exp\left(\int_{T_1}^{T_2}\sqrt{\zeta_s}\,dB_s - \tfrac{1}{2}\int_{T_1}^{T_2}\zeta_s ds\right) - k\right)^+$$

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Realized Variance
Beyond Variance Swaps

- Very popular: bets on *volatilty of volatility* (VolOfVol) in the form of straddles on realized variance:

$$\left| \int_0^T \zeta_s ds - K^2 \right|$$

  - The strike K is the variance swap strike (“ATM straddle”)
  - Initial “vol-delta” is zero, but high gamma.

- Also sought after: capped calls

$$\min\left\{ (20\% + K)^2, \left(\int_0^T \zeta_s ds - K^2 \right)^+ \right\}$$

Realized Variance

Modelling volatility

- How can we develop a model to price & hedge such payoffs?
- The “perfect” model for *all* single-underlying equity products would be a *stochastic implied volatility model*, where the evolution of the implied volatility surface $\sigma^{T,K}$ is directly described by a low-factor SDE (Brace et al 2001, Cont et al 2002)
  - Basic idea is to write (in one parametrization or the other)

$$d\sigma_t^{T,K} = A_t(\sigma_t^{T,K};T,K)dt + \sum_{j=1,\ldots,d} B_t^j(\sigma_t^{T,K};T,K)dW_t^j$$

  - If such a model is given, the all European option prices at all times are known.
  - Hence, all variance swap prices are known.
  - The stock price is the value of the just maturing zero-strike call.

Realized Variance
Modelling volatility

- Unfortunately, all known applications suffer from severe problems.
  1. It is surprisingly complicated to “design” an implied volatilty parametrization which is truly free of arbitrage (no negative butterflies, no negative calendar spreads and boundary conditions).
     – This is essential to guarantee absence of “static” arbitrage.
  2. Even given such a surface, its dynamics cannot be freely chosen: to ensure that the prices of European options are at least local martingales, we have to impose quite complicated constraints on the drift and volatility coefficients of the SDE for the implied volatility.
     – This is necessary to guarantee absence of “dynamic” arbitrage”.
- Other approaches in a similar vein are to model the implied local volatility surface of the stock, its implied density or simply the European option price surface directly.
  $\rightarrow$ Similar problems arise

Realized Variance
Modelling volatility

- However, in case of “option on variance”, intuitively it should be sufficient to model only the variance swaps: the idea is that variance swaps can be used to “delta-hedge” more complex options on realized variance.
  – Of course, to obtain a useful model, we will also want to model the stock price itself and to develop a good concept of “skew”.

- Mathematically, the term-structure of variance swaps reminds on the term-structure of discount bounds in interest rate models.
  – It is therefore tempting to apply concepts from interest rate theory to the pricing of options on variance.
  – A similar idea has been discussed here recently by Bergomi (2005)

Variance Curve Models

Modelling volatility

Variance Curve Models

Program

- Instead of starting with S as in classic stochastic volatility models, let us first specify the dynamics of the variance swaps .
- Then, construct a (local) martingale S which has the correct quadratic variation.
- The correlation between the Brownian motion which drives S and the variance curve will function as a skew parameter.
- The model shall also yield hedging ratios in terms of variance swaps; we discuss details on hedging in Markovian models.

Variance Curve Models

Forward Variance

- Variance swap prices are increasing with maturity T.
  - Their price at a later time t also depends on the past realized variance.
- To alleviate these unpleasant properties, note that

$$V_t(T) = \mathrm{E}\left[\int_0^T \zeta_s ds \mid \mathrm{F}_t\right] = \int_0^T \mathrm{E}\left[\zeta_s \mid \mathrm{F}_t\right] ds$$

can be differentiated in T to define the *forward variance*

$$v_t(T) := \partial_T V_t(T) = \mathrm{E}\left[\zeta_T \mid \mathrm{F}_t\right]$$

Observation time **Maturity**

- Note the similarity to the *forward rate* in interest rate theory.
- An important property is that forward variance can be zero.

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Variance Curve Models

Classic approach

- Assume we have a driving d-dimensional extremal Brownian motion W on the space $(\Omega,\mathrm{P},\mathrm{F})$.

- Definition
  A family $v=(v(T))_{T\geq 0}$ is called a [local] *Variance Curve Model* if

1. For each $T>0$, the process $v(T)=(v_t(T))_{t\in[0,T]}$ is a non-negative [local] martingale:

$$dv_t(T) = \sum_{j=1,\ldots,d} \beta_t^j(T)dW_t^j \qquad \beta^j(T) \in L^{\mathrm{loc}}$$

2. For each $T>0$, the initial variance swap prices are finite, i.e.

$$V_0(T) = \int_0^T v_0(s)ds < \infty$$

3. The curve $v_t(t)$ is left-continuous.

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Variance Curve Models

Classic approach

- Properties
  - The price processes of variance swaps,

$$V_t(T) := \int_0^T v_t(s)ds$$

    are [local] martingales.

  - The *short variance process*

$$\zeta_t := v_t(t)$$

    is well defined, integrable and non-negative.

Variance Curve Models

Classic approach

- Properties
  Given any standard Brownian motion B on $(\Omega, \mathrm{P}, \mathrm{F})$, the process

$$dX_t = \sqrt{\zeta_t}\, dB_t$$

is a square-integrable martingale, so the via B *associated stock price*

$$S_t := \exp\left(X_t - \tfrac{1}{2}\langle X\rangle_t\right)$$

is a local martingale.

  – B represents the *correlation structure* of S with v.

- Theorem
  For each variance curve model v and each Brownian motion B, the market

$$\left(S; (V(T))_{T\geq 0}\right)$$

is free of arbitrage.

Variance Curve Models

Classic approach – Musiela-Parametrization

- As in interest rates, it is more convenient to work with fixed time-to-maturities $x:=T-t$. Hence we define the *Musiela parameterization*

$$u_t(x) := v_t(t+x)$$

- Starting in Musiela-parametrization
  - Assume that $\sum_{j=1,\dots,d} \int_0^\infty \int_t^\infty \partial_T \beta_t(T)^2 \, dT dt < \infty$
    Then,

$$du_t(x) := \partial_x u_t(x) dt + \sum_{j=1,\dots,d} b_t^j(x) dW_t^j$$

defines a local variance curve model in Musiela-parametrization.

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Variance Curve Models

Classic approach – Fitting the market

- If v is represented as an exponential,

$$u_t(x) := \exp(w_t(x))$$

  it allows us to fit the model easily to an observed market forward variance curve m_0 by setting $w_0 := \log m_0$ (cf. Dupire, 2004).

  – This construction does not allow u to become zero and therefore excludes classic stochastic volatility model such as Heston's.

- This approach can be extended to any given model u^{base} by setting

$$u_t(x) := \frac{m_0(t+x)}{u_0^{\text{base}}(t+x)} u_t^{\text{base}}(x)$$

  – Mind effects on the martingale property of S.

Variance Curve Models

Variance Curve Functionals

- Problems with a specification with general integrands $b(T)$:
  - It is complicated to check whether u remains non-negative.
  - In practice, it is not clear how to handle such integrands computationally.
- Ideally, we want to write

$$u_t(x) := G(Z_t; x)$$

  for some suitable non-negative function G and an m-dimensional Markov-process Z.

- The function is the "interpolation function" for the forward variances.

Variance Curve Models

Variance Curve Functionals

- Definition

1. A non-negative $C^{2,2}$-function $G: D x R^+ \rightarrow R^+$ is called a *Variance Curve Functional* if

$$\int_0^T G(z;x)dx < \infty$$

for all T and $z \in D$ where D is an open set in $R_{\geq 0}^m$.

2. We denote by Ξ the set of all $C=(\mu,\sigma)$ for which the SDE

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\ldots,d} \sigma^j(Z_t)dW_t^j$$

starting at any point $Z_0 \in D$ has a unique solution Z which stays in D.

   – Time-dependency is included in this setup.

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Variance Curve Models

Variance Curve Functionals

- Definition
  We call $C=(\mu,\sigma)\in\Xi$ a *consistent factor model* for G if for any $Z_0 \in D$,

$$u_t(x) := G(Z_t; x)$$

  defines a local variance curve model.

- Theorem
  This is the case if and only if Z stays in D and if

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^T \sigma(z)\,\partial^2_{zz} G(z;x)$$

  holds.

Variance Curve Models

Variance Curve Functionals

- Local Correlation and the Markov property
  Given a consistent factor model $C=(\mu,\sigma)\in\Xi$ and a "correlation function" $\rho:R^{+}xD\rightarrow[-1,1]^{d}$ with $|\rho|_2=1$, we can always define

$$dS_t = \sum_{j=1,\ldots,d} S_t \rho^j(S_t;Z_t)\left\{\sqrt{G(Z_t;0)}dW_t^j\right\}$$

  such that the process (S,Z) is Markov and S is a local martingale (note that the SDE does not explode).

  – Local-Stochastic volatility "mixture models" are also part of this framework: they correspond to the case where S is one of the factors of Z.

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Variance Curve Models

Variance Curve Functionals

- <u>Market completeness</u>
  Assume that
  - (S,Z) is Markov as on the previous slide.
  - The variance swap price function

$$VG(z,x) := \int_0^x G(z,y)dy$$

is invertible in an appropriate sense (cf. Buehler (2006)).

  - For all smooth functions f whose derivatives all have compact support, the function

$$P_t f(s,z) := \mathrm{E}\left[f(S_t, Z_t) \mid S_t = s, Z_t = z \right]$$

is once differentiable.

Then, the market of "relevant payoffs" is complete (i.e. those payoffs which depend only on the price history of S and the variance swaps).

Variance Curve Models

Term-structure approach

- The next logical step is to model the entire curve u as a process with values in a Hilbert space H.
  - The follows the path laid by Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004) and Teichmann (2005).

- We omit this discussion here and refer to Buehler (2006).

Variance Curve Models

Term-structure approach

- The next logical step is to model the entire curve u as a process with values in a Hilbert space H.
  - We follow the path laid by Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004) and Teichmann (2005).

- The main difference between variance curves and forward curves is that the curves u must remain non-negative (but *can* become zero).
  - The problem is that the “non-negative cone” is a very small set. Indeed it has no interior points.
  - However, if $G(D)$ is a sub-manifold with boundary of H, then it is sufficient to check whether u stays in $G(D)$. In this case we say $G(D)$ is *locally invariant* for u.
  - If G is moreover invertible, we can also directly construct a (locally) consistent factor model $C=(\mu,\sigma)$ for G.

Variance Curve Models

Term-structure approach

- Assume that the variance curve u is given as a solution in H to

$$du_t = \partial_x u_t dt + \sum_{j=1,\ldots,d} b^j(u_t) dW_t^j$$

where the coefficients β are locally Lipschitz vector fields.

- The Stratonovic-drift for u is as usual

Frechet-Derivative

$$\beta^0(u) := \partial_x u - \sum_{j=1,\ldots,d} D\beta^j(u) \cdot \beta^j(u)$$

such that

$$du_t = \hat{\beta}^0(u_t)dt + \sum_{j=1,\ldots,d} \hat{\beta}^j(u_t) \circ dW_t^j$$

Stratonovic-Integral

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Variance Curve Models

Term-structure approach

- Theorem (Filipovic/Teichmann 2004)
  The sub-manifold $G(D)$ is locally invariant for u iff

  1. We have $G(D) \subset \mathrm{dom}(\partial x)$,
  2. In the interior of $G(D)$, we have

$$\hat{\beta}^{j}(u) \in T_u G(D) \qquad j = 0, \ldots, d$$

  3. On the boundary $\partial G(D)$,

$$\hat{\beta}^{0}(u) \in (T_u G(D))_{\geq 0}$$

$$\hat{\beta}^{j}(u) \in T_u \partial G(D) \qquad j = 1, \ldots, d$$

holds.

Variance Curve Models

Term-structure approach

- If we can invert G, then $C=(\mu,\sigma)$ with

$$\sigma^j(z) := \partial_z G^{-1}(\hat{\beta}^j(G(z))$$

$$\mu(z) := \partial_z G^{-1}(\hat{\beta}^0(G(z)) + \sum_{j=1,..d} (\partial_z \sigma^j)(z)\sigma^j(z)$$

is a consistent factor model for G.

Back to reality: Applications

How to model variance curves

Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- Example
  A very basic example is the “linearly mean-reverting” functional:

„Speed of mean-reversion“

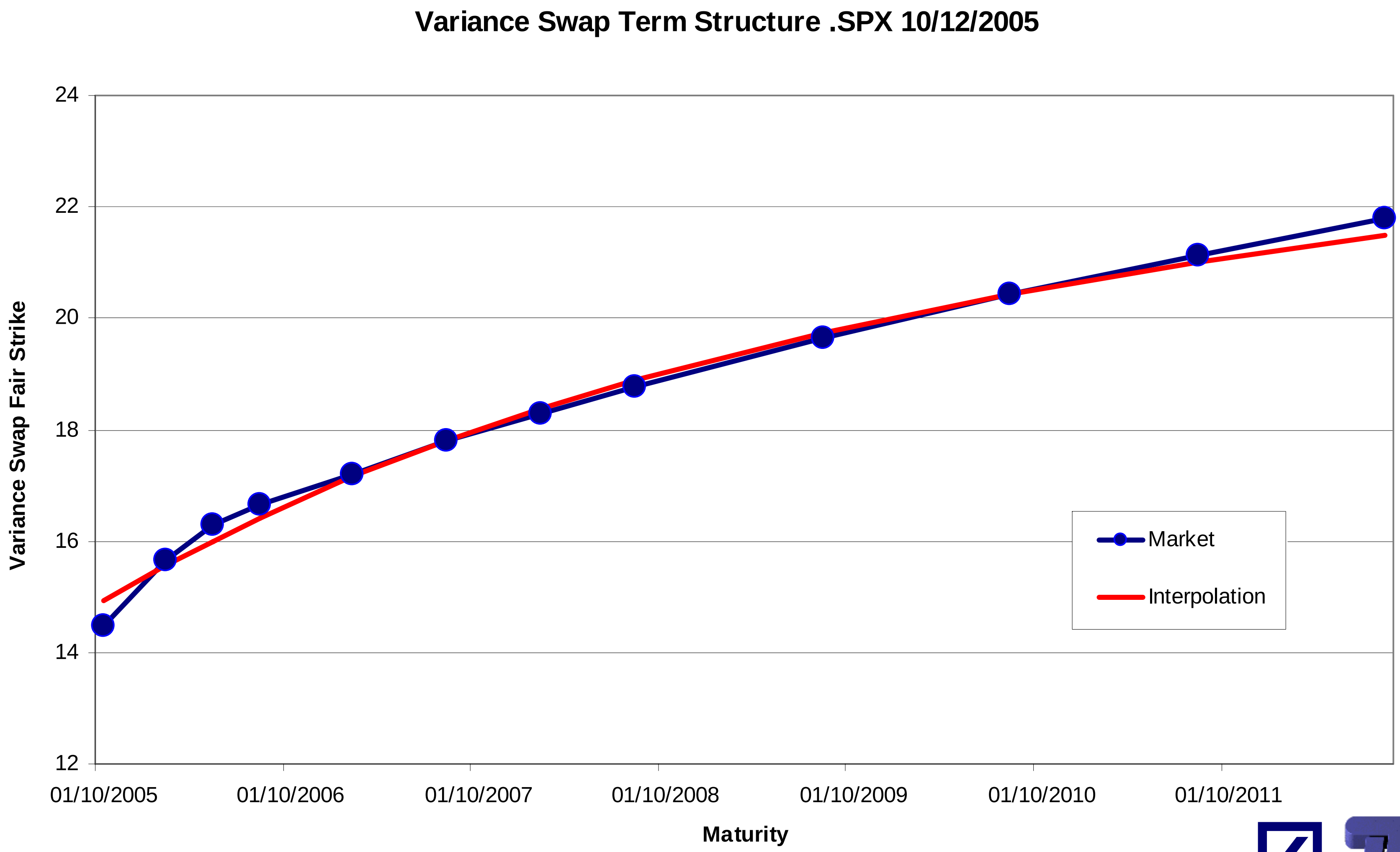
$$G(z;x) = z_2 + (z_1 - z_2)e^{-z_3 x}$$

„Long variance“ „Short variance“

for $z_1 \geq 0$ and $z_2, z_3 > 0$.

Variance Curve Models

Variance Curve Functionals – Linear mean-reversion



Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- Question: What dynamics can a consistent process $Z=(Z_1, Z_2, Z_3)$ have?
- The coefficients μ and σ have to satisfy

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^T\sigma(z)\partial^2_{zz}G(z;x)$$

1. First, we see that
$$\partial^2_{z_3 z_3} G(z;x) = (z_1 - z_2)x^2 e^{-z_3 x}$$
Since no term x^2e^x appears on the left hand side, we must have $\sigma_3=0$.
2. The same line of thought applied to
$$\partial_{z_3} G(z,x) = -(z_1 - z_2)xe^{-z_3 x}$$
shows that we also have $\mu_3=0$.

Hence, the speed of mean-reversion cannot be stochastic.

Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- For the other two parameters, we find that while σ is unconstrained,

$$\mu_2(z) = 0$$
$$\mu_1(z) = z_3(z_2 - z_1)$$

In other words: the only consistent processes for this choice of G are of Heston-type

Linear mean-reversion drift

$$d\zeta_t = \kappa(\theta_t - \zeta_t)dt + \sigma_1(\zeta_t, \theta_t)dW_t$$
$$d\theta_t = \sigma_2(\zeta_t, \theta_t)dW_t$$

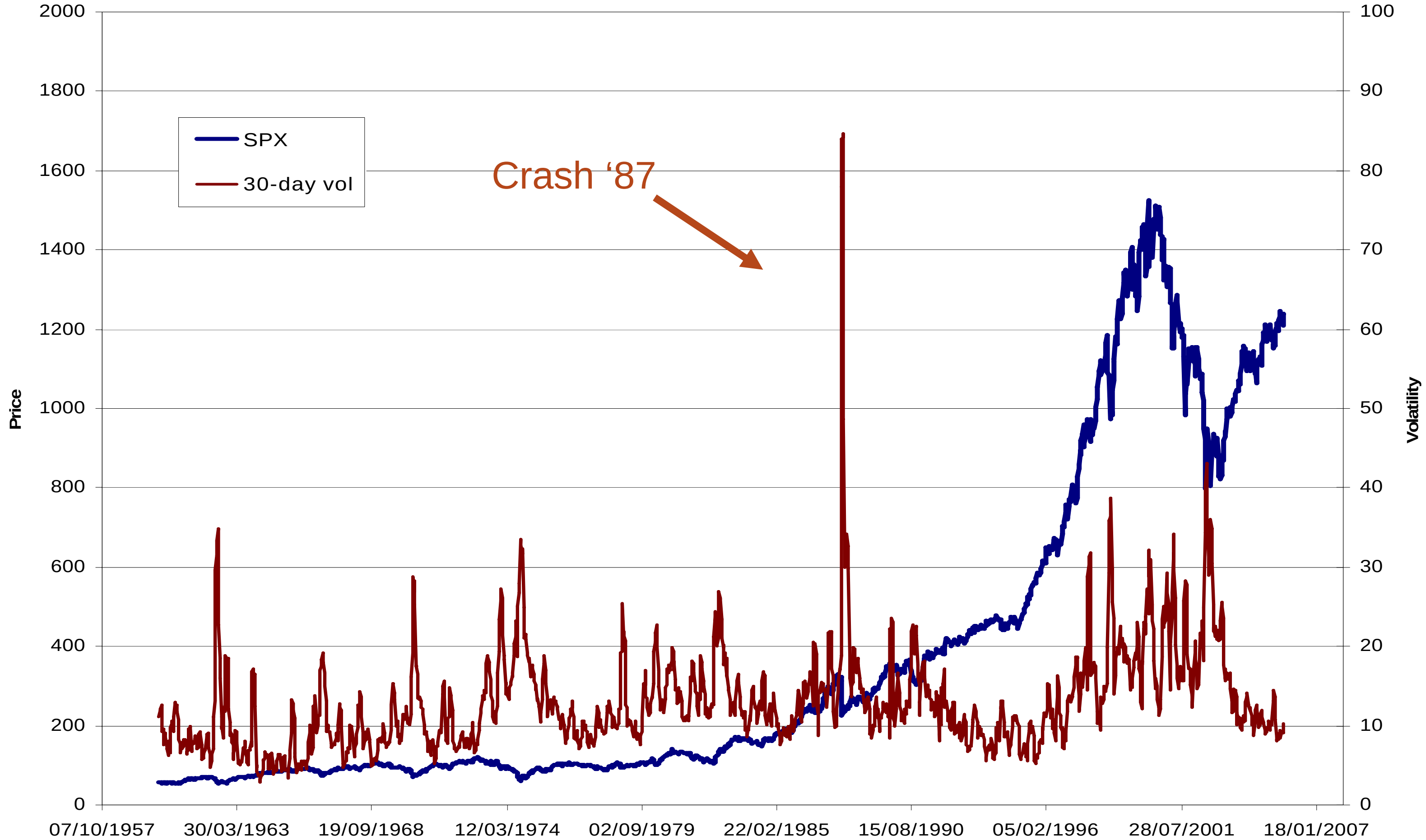
VolOfVol can freely be chosen as long as ζ remains non-negative.

Mean-reversion level θ is a positive martingale.

Variance Curve Models

Why mean-reversion?

SPX Spot level and 30-day realized volatility

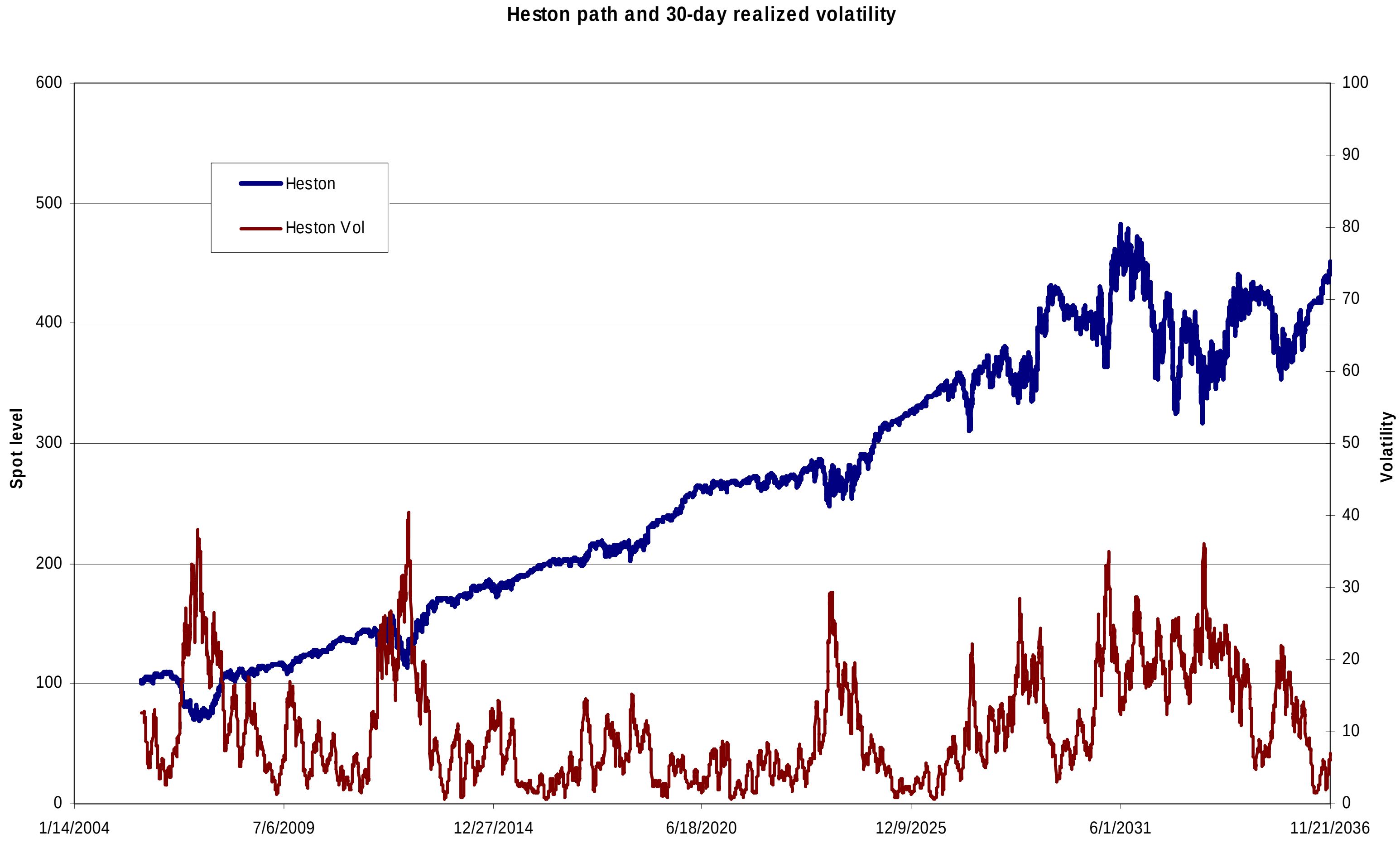


Variance Curve Models

Why mean-reversion?

Unconstrained Calibration

ShortVol	14.4%
LongVol	28.7%
RevSpeed	0.23
Correlation	-0.74
VolOfVol	26.3%



Variance Curve Models

Variance Curve Functionals

- Proposition
  The observation that mean-reversion speeds must be constant holds for all polynomial-exponential functionals, i.e. if $(p_i)_i$ are polynomials

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\sum_{i=1}^{n} p_i(z;x)e^{-z_i x}$$

  then the first n components must be constant (cf. Filipovic 2001 for interest rates).

- A similarly restrictive result can be shown for functionals of the form

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\exp\left\{\sum_{i=1}^{n} p_i(z;x)e^{-z_i x}\right\}$$

  - The parameters in the exponent come in pairs, where one is twice as large as the other (again Filipovic 2001).

Variance Curve Models

Variance Curve Functionals – "Double Heston"

- Another example of the polynomial-exponential class is

$$G(z;x) = z_3 + (z_1 - z_2)e^{-\kappa x} + (z_2 - z_3)\frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right)$$

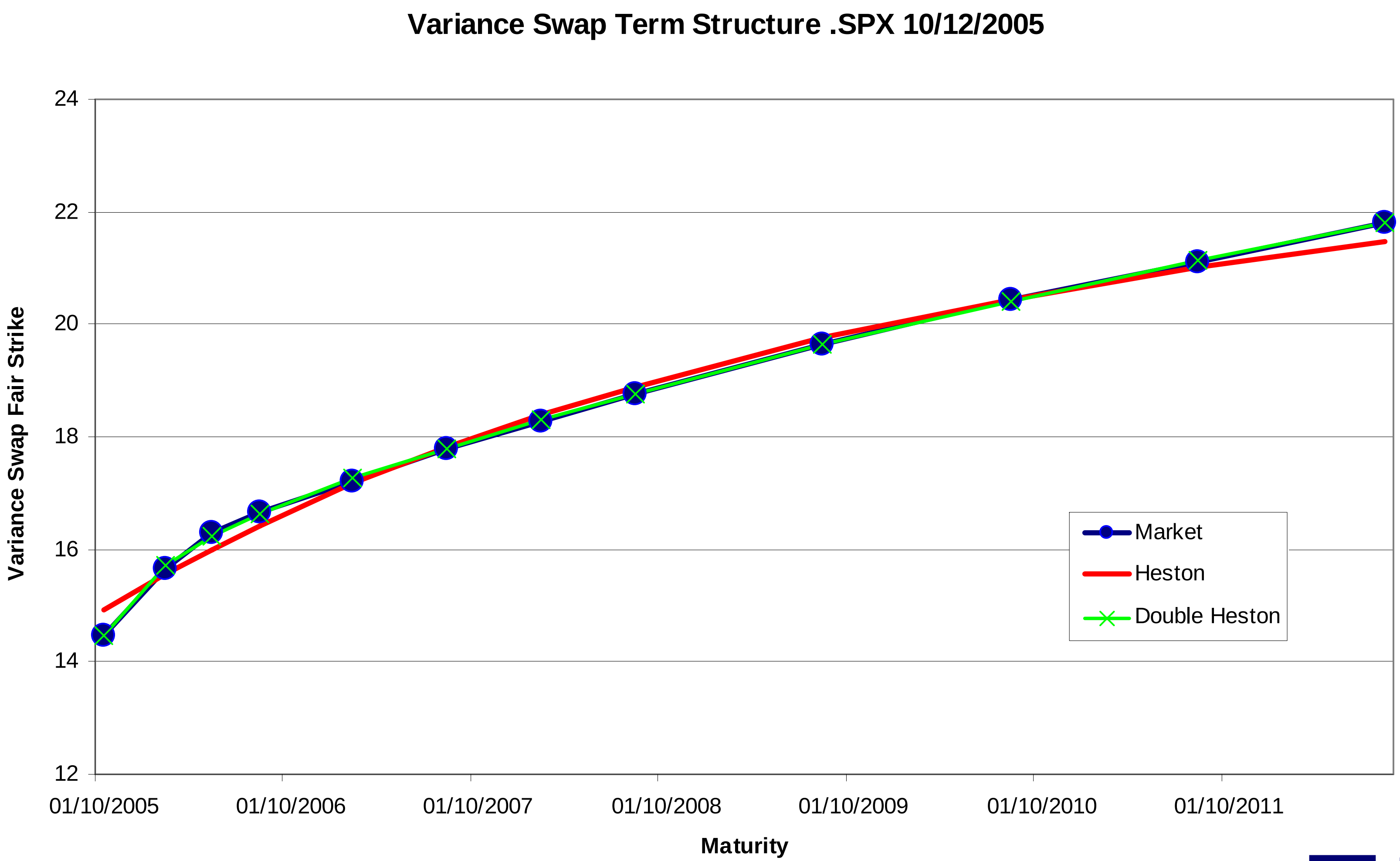
  - A consistent factor model for this G must have the form

$$\begin{aligned} dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \sigma_1(Z_t)dW_t \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \sigma_2(Z_t)dW_t \\ dZ_t^3 &= \sigma_3(Z_t)dW_t \end{aligned}$$

    which we call "double mean-reverting".
  - Quite a good fit for most indices (at least during the course of the last year).
  - This is in effect Svensson's interpolation function for interest rates.

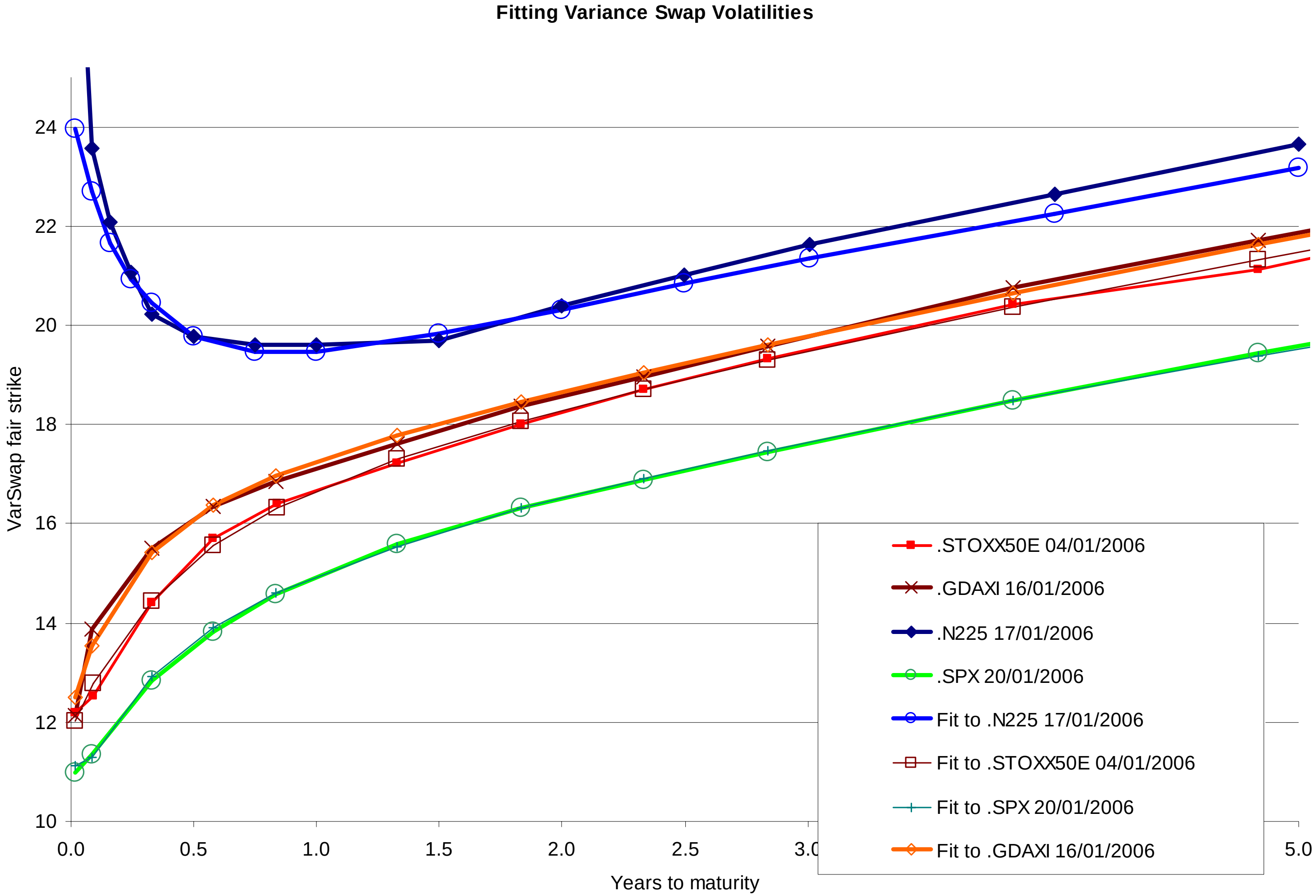
Variance Curve Models

Variance Curve Functionals – “Double Heston”



Variance Curve Models

Variance Curve Functionals – “Double Heston”



Variance Curve Models

Variance Curve Functionals – "Double Heston"

- Such a model is discussed in "Equity Hybrid Derivatives" (2006) where we used

$$
\begin{aligned}
d\zeta_t &= \kappa(\theta_t - \zeta_t)dt + \nu\zeta_t^{\alpha} d\hat{W}_t^1 \\
d\theta_t &= c(m_t - \theta_t)dt + \mu\theta_t^{\beta} d\hat{W}_t^2 \\
dm_t &= \eta m_t d\hat{W}_t^3
\end{aligned}
$$

  for a correlated vector of Brownian motions (½ < $\alpha,\beta \leq$ 1).

  – To calibrate it, we first fit the variance curve function itself.

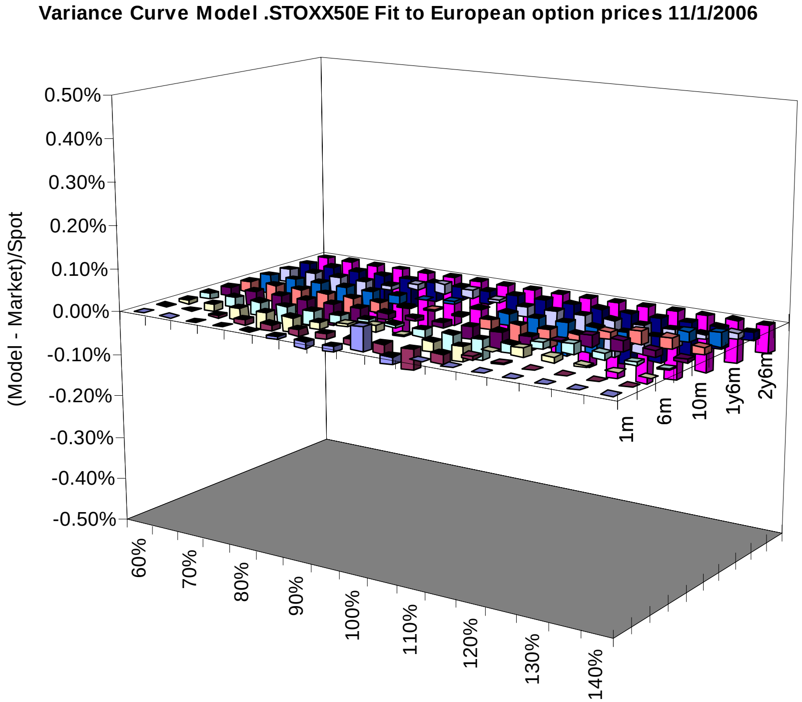
  – In a second step, we use European option prices close to ATM to calibrate the volatility and correlation parameters.

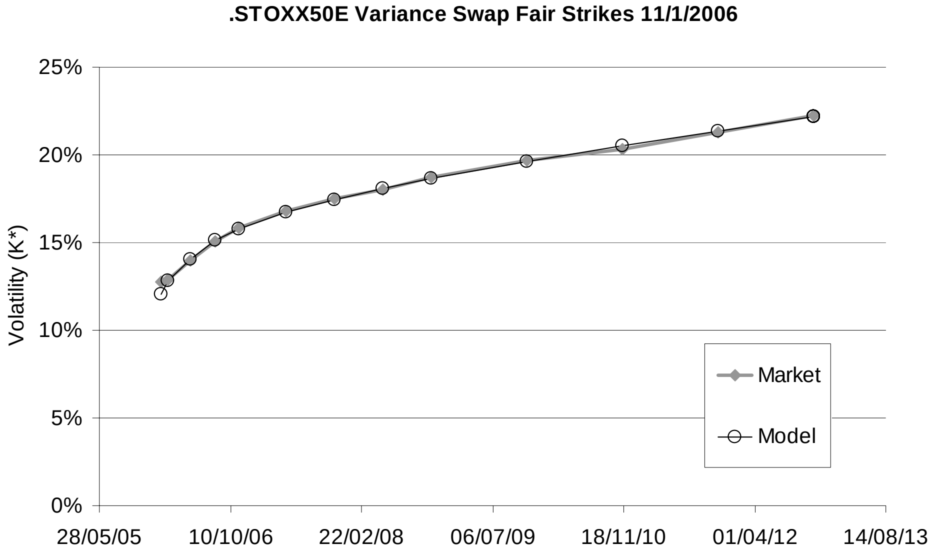
  – Numerically quite tedious.

- The stock price in this model is a martingale if the correlations between the BM which drives the stock and the BMs above are all non positive.

Variance Curve Models

Variance Curve Functionals – “Double Heston”





Variance Curve Models

Variance Curve Functionals – “Double Heston Lite”

- Intuitively, a more-factor model is only necessary if we want to price options on variance swaps etc ($\rightarrow$ later)
- A model with the same variance swap term structure, but which is much easier to handle numerically is

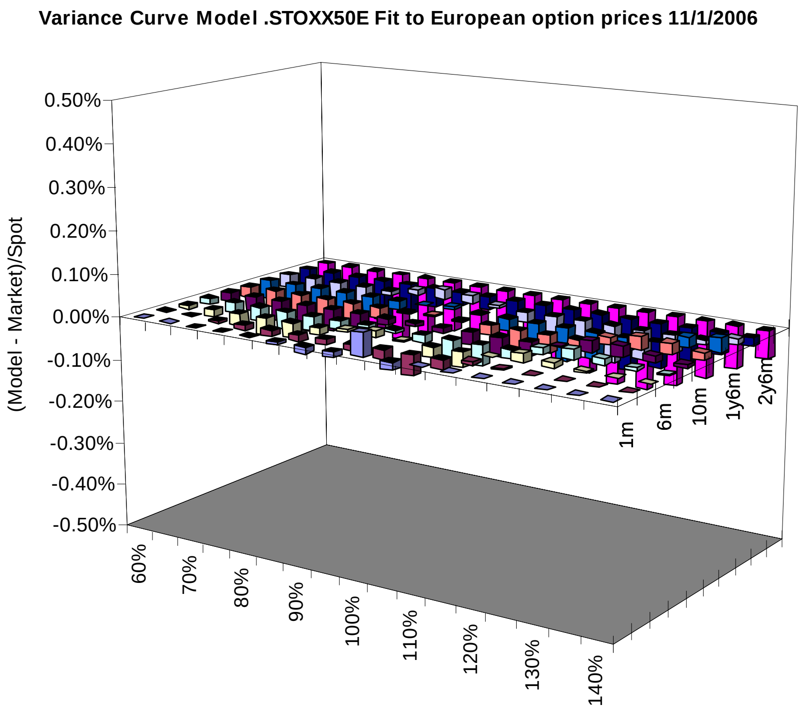
$$\begin{aligned} d\zeta_t &= \kappa(\theta_t - \zeta_t)dt + \nu\sqrt{\zeta_t}\,d\hat{W}_t^1 \\ d\theta_t &= c(m - \theta_t)dt \end{aligned}$$

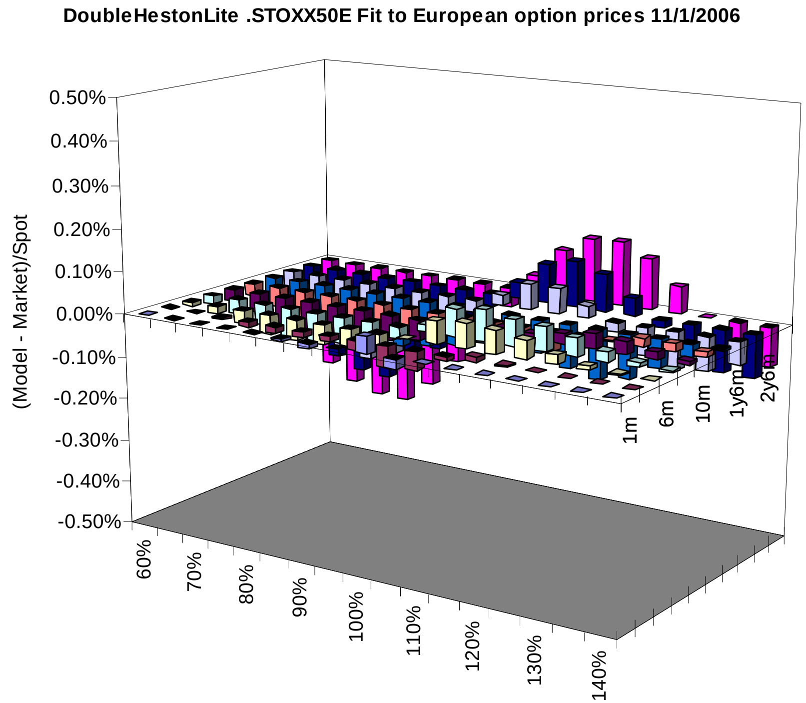
with piece-wise constant VolOfVol and correlation between stock and variance.

  – European options on the stock can be priced using Fourier Inversion.
  – Stock price is a martingale.
  – But mind square integrability in this and the previous model.

Variance Curve Models

Variance Curve Functionals – Impact of Multi-factor models





The two models have the same initial variance swap term structure.

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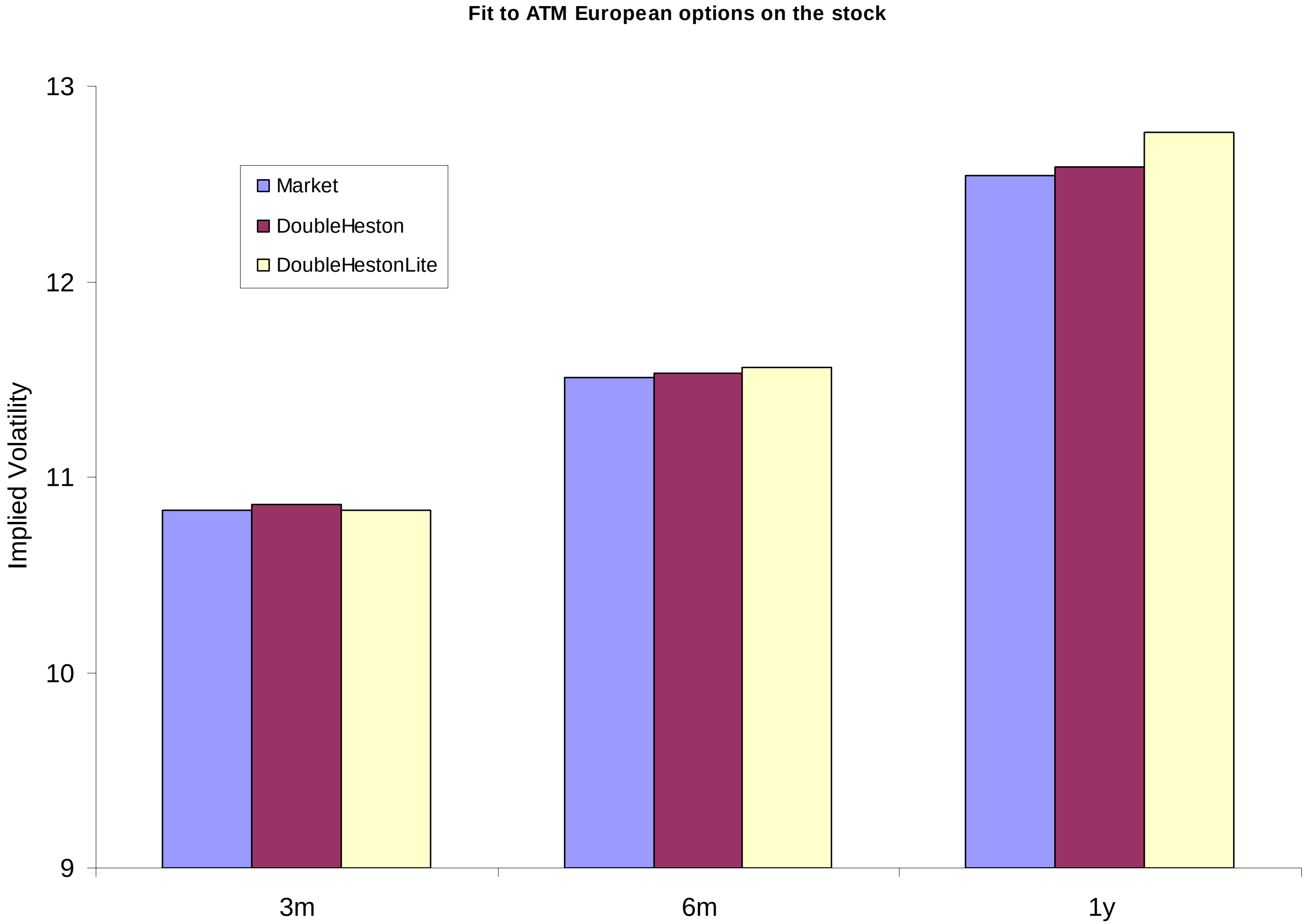
Variance Curve Models

Variance Curve Functionals – Impact of Multi-factor models



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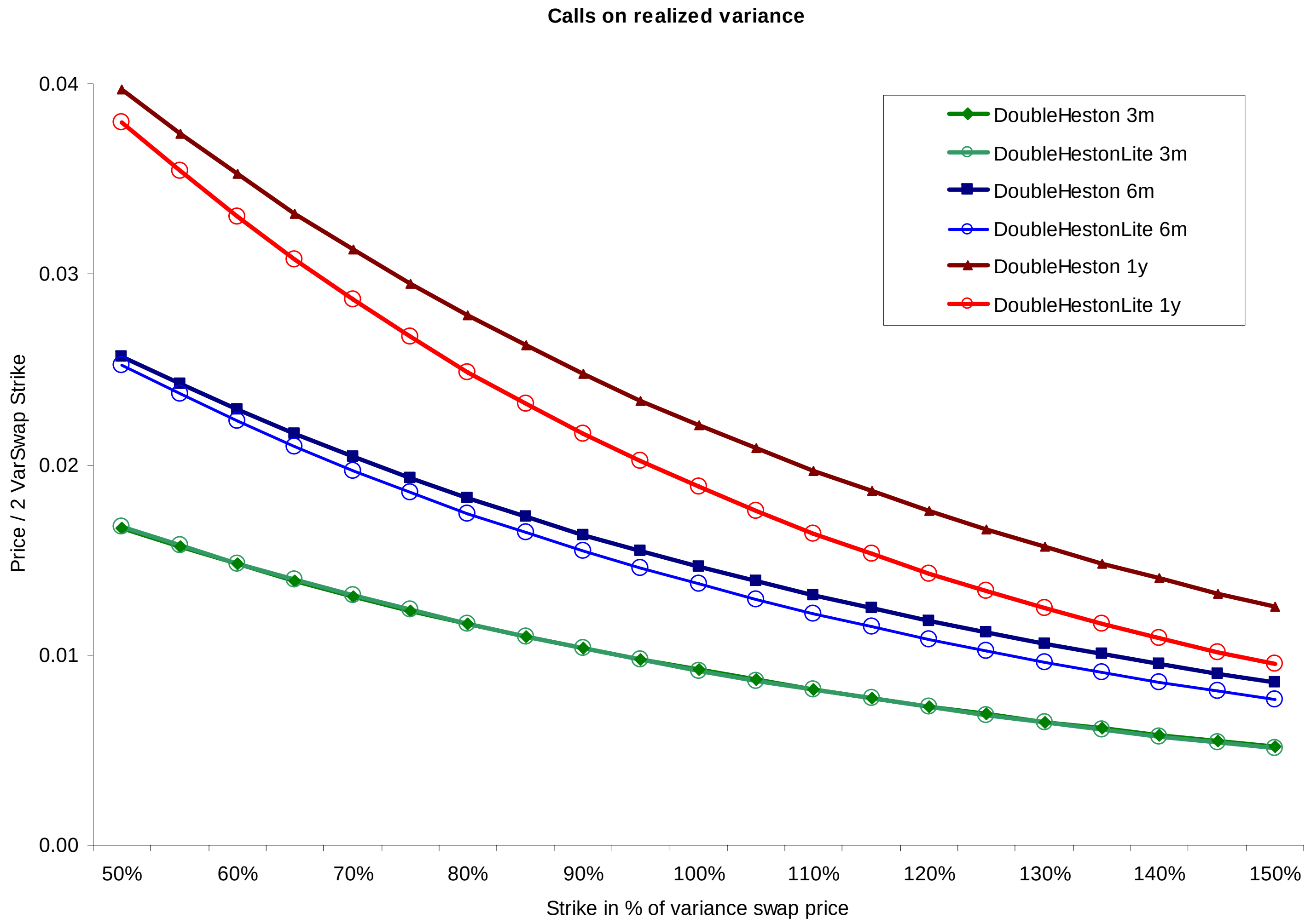
Variance Curve Models

Variance Curve Functionals – Impact of Multi-factor models



The two models have the same initial variance swap term structure.

Variance Curve Models

Variance Curve Functionals – Impact of Multi-factor models

- To assess the impact of using a multi-factor model to pricing more exotic options on variance, we now recalibrate the "VolOfVol" of the one-factor model such that we match the 1y ATM option on realized variance.
  - We then price forward started options with reset date 1y and maturity 2y

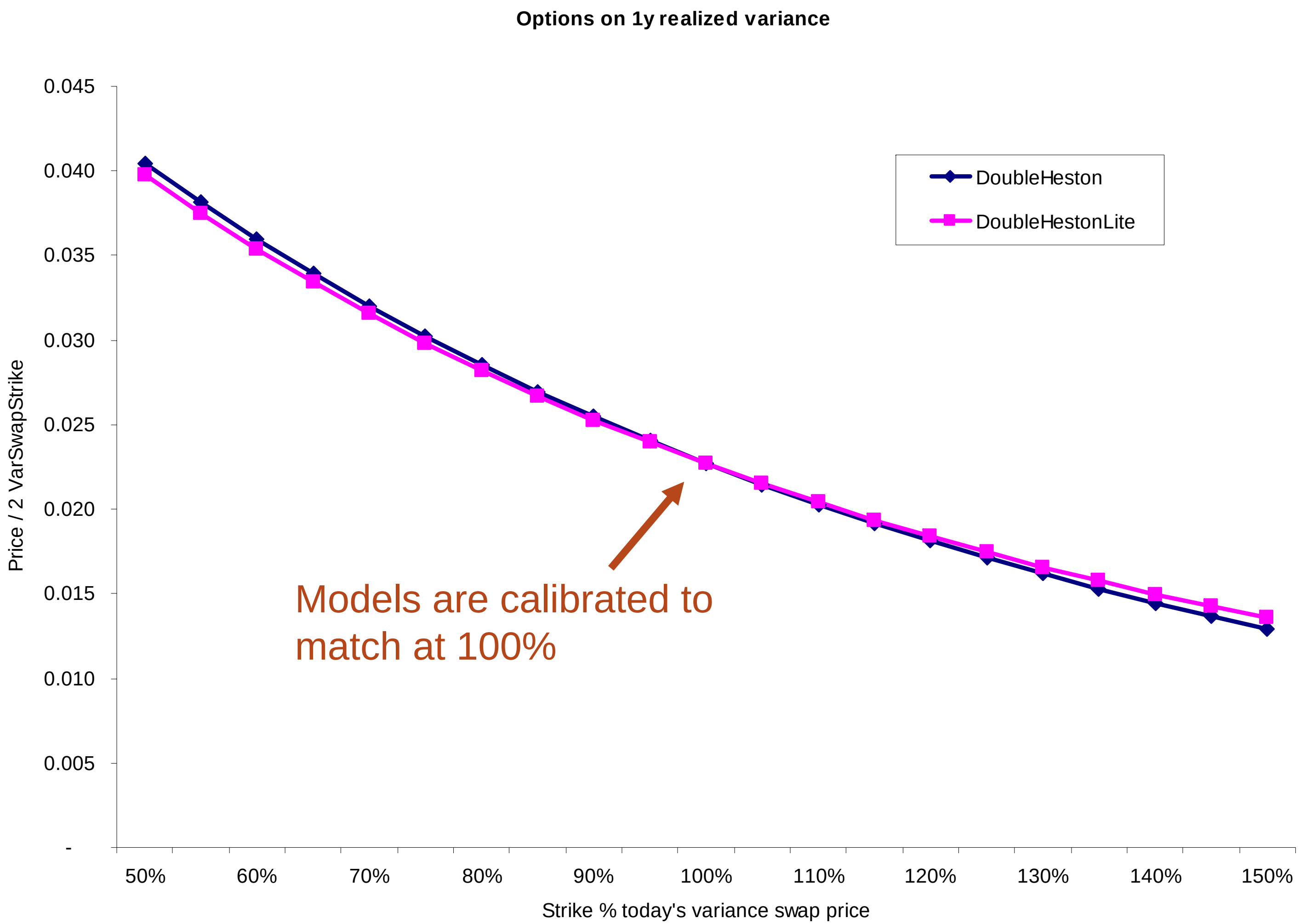
$$\left(\int_{T_R}^{T} \zeta_t dt - kV_{T_R}(T_R,T)\right)^+ \Big/ 2(T-T_R)\sqrt{V_0(T_R,T)}$$

  - We also price options on the 1y to 2y variance swap

$$\left(V_t(T_R,T) - kV_0(T_R,T)\right)^+ \Big/ 2(T-T_R)\sqrt{V_0(T_R,T)}$$

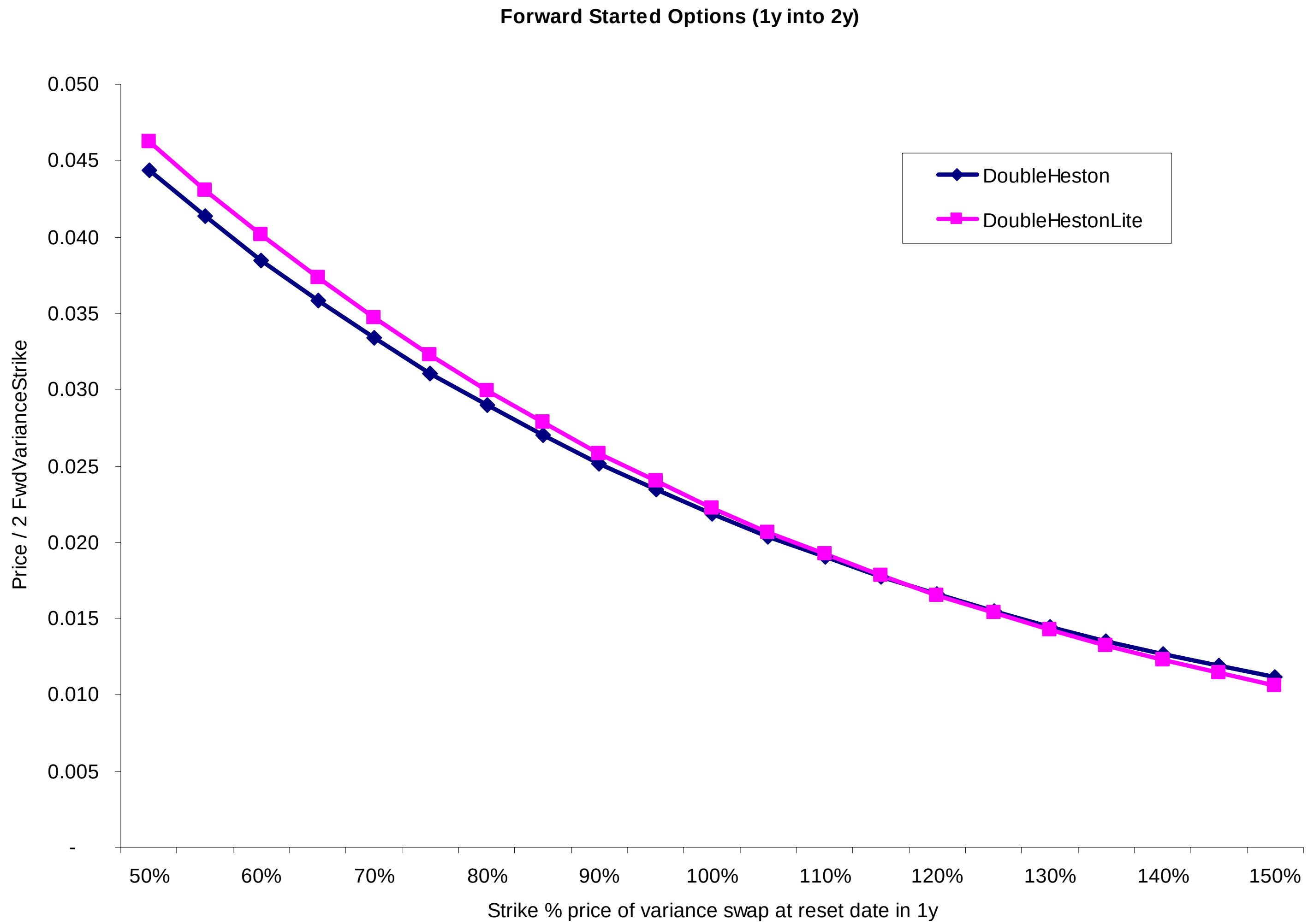
Variance Curve Models

Variance Curve Functionals – Impact of Multi-factor models



Variance Curve Models

Variance Curve Functionals – Impact of Multi-factor models



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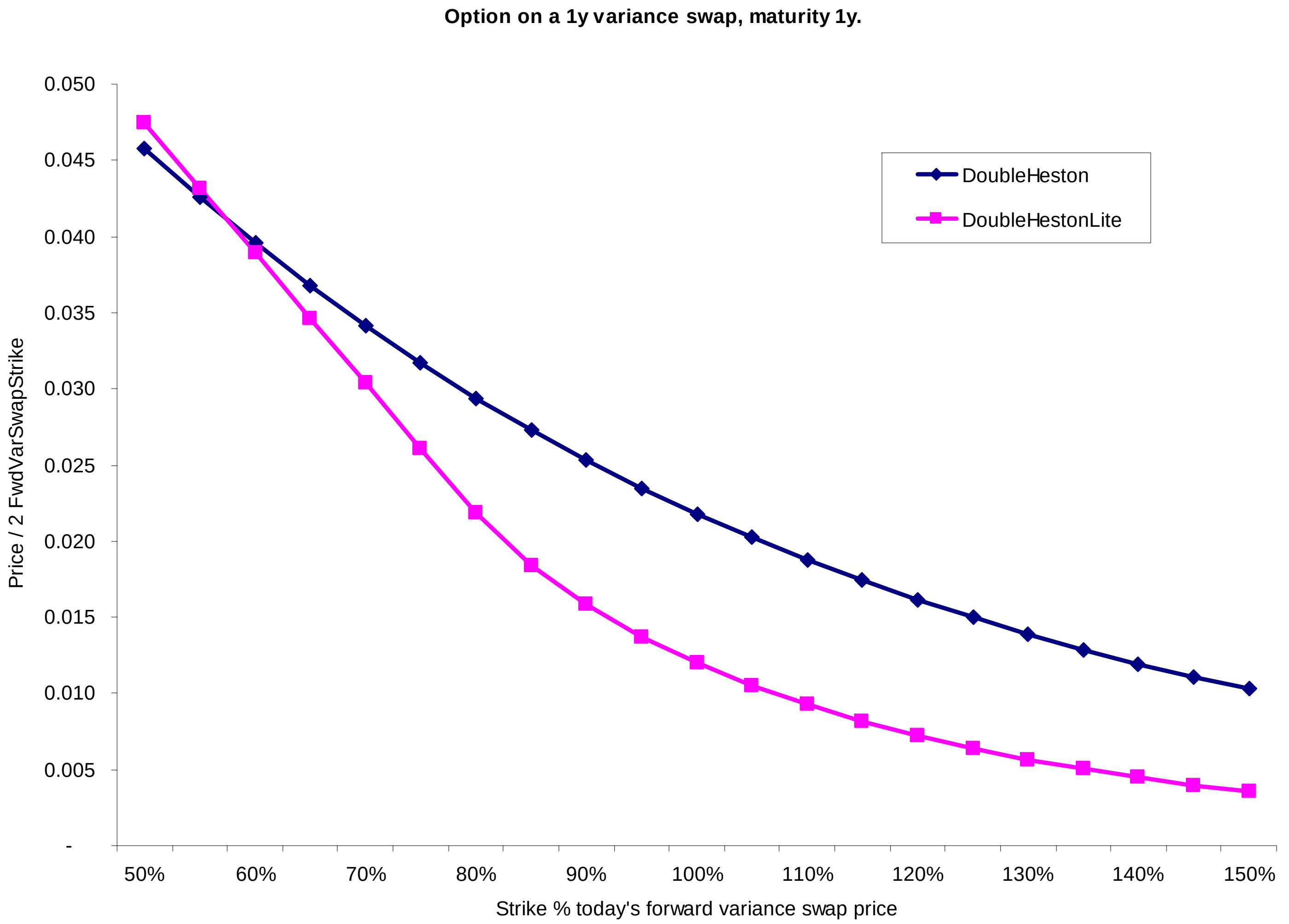
Variance Curve Models

Variance Curve Functionals – Impact of Multi-factor models



Fitting the market with one-factor models

A case study

Fit the market

Using variance curve models in practise

- To price vanilla options on realized variance, it is sufficient to use a one-factor model which we fit to an observed market curve m_0.
  - The calibration of the one-factor model to the multi-factor model has shown that both feature very similar prices once matched for the ATM option.
- <u>Example 1:</u>
  Fitted Log-Normal (for $\kappa=0$ we obtain Dupire 2004, two-factor version discussed by Bergomi 2005)

$$\zeta_t := m_0(t)\frac{e^{\hat{u}_t}}{\mathrm{E}\left[e^{\hat{u}_t}\right]} \qquad d\hat{u}_t = -\kappa\hat{u}_t dt + \sigma dW_t$$

  - No closed form for European options.

Fit the market

Using variance curve models in practise

- Fitted Heston model

$$d\zeta_t = \kappa\left(\theta(t) - \zeta_t\right)dt + \nu\sqrt{\zeta_t}\,dW_t$$

  - We set $\theta(t) := \kappa m_0(t) + \partial_t m_0(t)$ which needs to remain non-negative to ensure that the process ζ is well-defined.
  - Martingale property of S preserved as long as correlation is not positive.
  - European options on the stock price can be priced reasonably quick using Fourier-Inversion.
    Piece-wise constant VolOfVol and Correlation parameters possible.

Fit the market
Using variance curve models in practise

- Another alternative, but not very pretty from a modelling point of view:
  - Use Heston's model for variance and stock and apply deterministic time-change to <u>both</u> variance and stock to fit the variance swap market:

$$\zeta_t := \hat{\zeta}_{m_0(t)/\hat{u}_0} \qquad d\hat{u}_t = \left(\hat{\zeta}_0 - \hat{\zeta}_t\right)dt + \nu\sqrt{\hat{\zeta}_t}\,dW_t$$

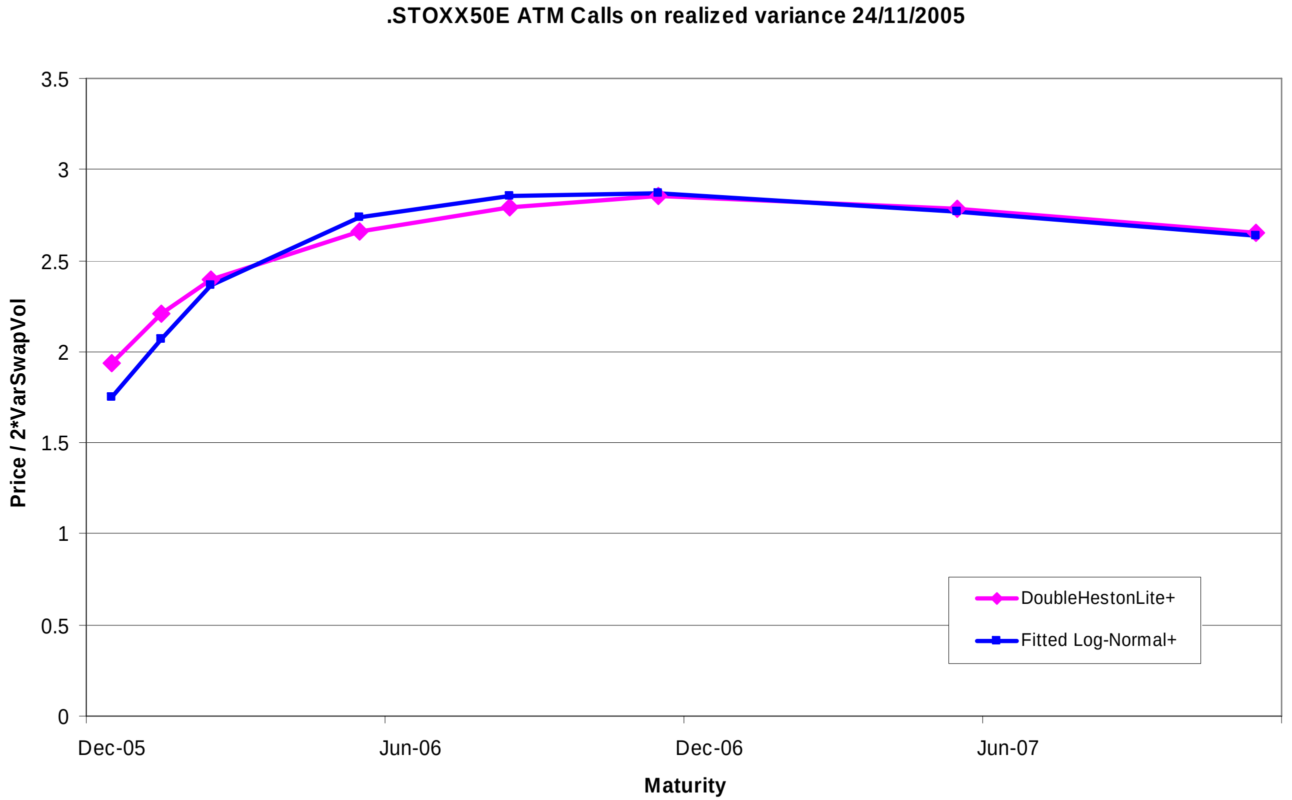
  - Again, pricing of European options on the stock is quick.
  - In terms of the variance process, changing time essentially means that

$$d\zeta_t = \kappa(t)\left(\hat{\zeta}_0 - \zeta_t\right)dt + \nu\sqrt{\kappa(t)\zeta_t}\,d\widetilde{W}_t$$

$$\kappa(t) = \partial_t\left(m_0(t)/\hat{u}_0\right)$$

  - Mind additional effects on the stock price dynamics.

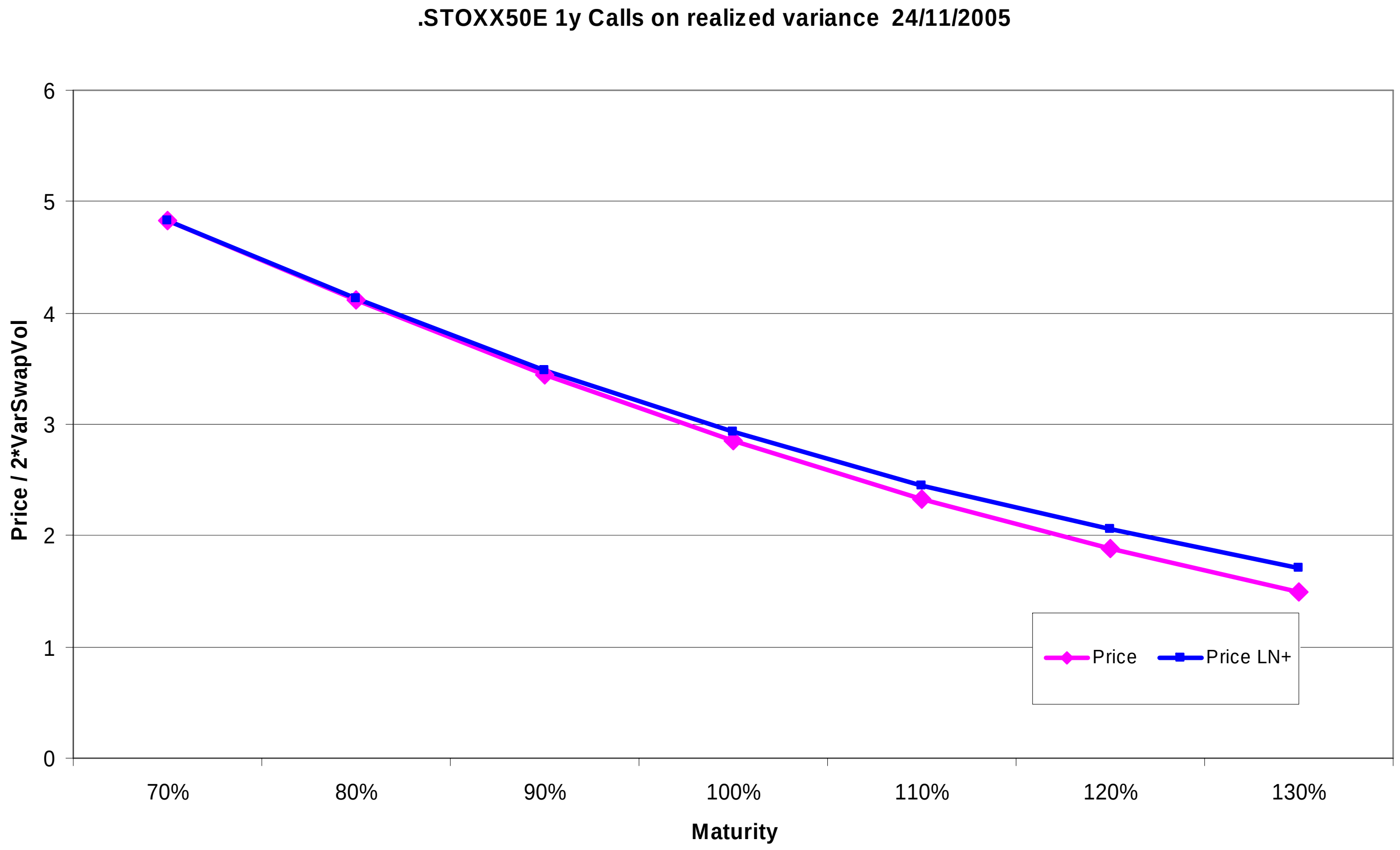
Fit the market
Fitting stochastic volatility models



Fitted Heston vol parameters are calibrated to term-structure of ATM equity options.
Fitted LN vol parameters are fitted by hand to the 1y and 2y call for both models.

Fit the market

Fitting stochastic volatility models



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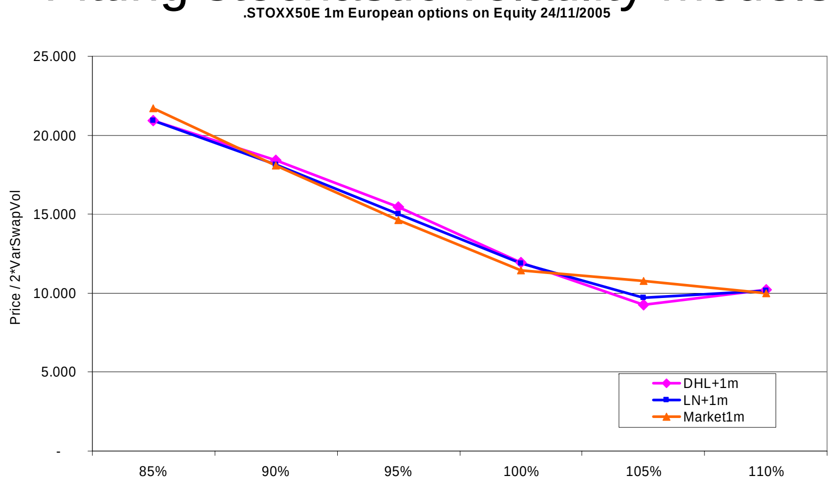
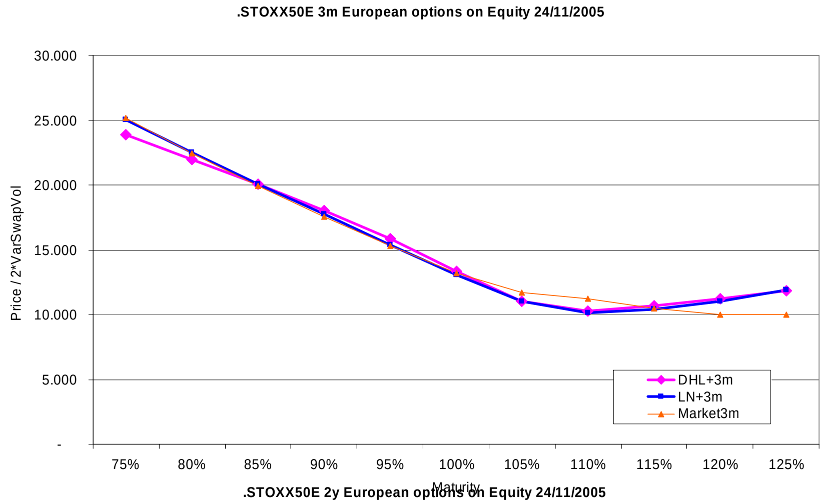
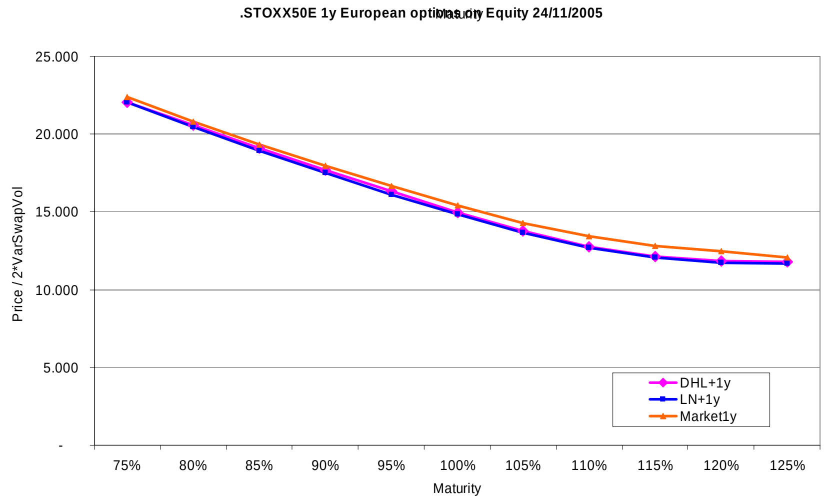
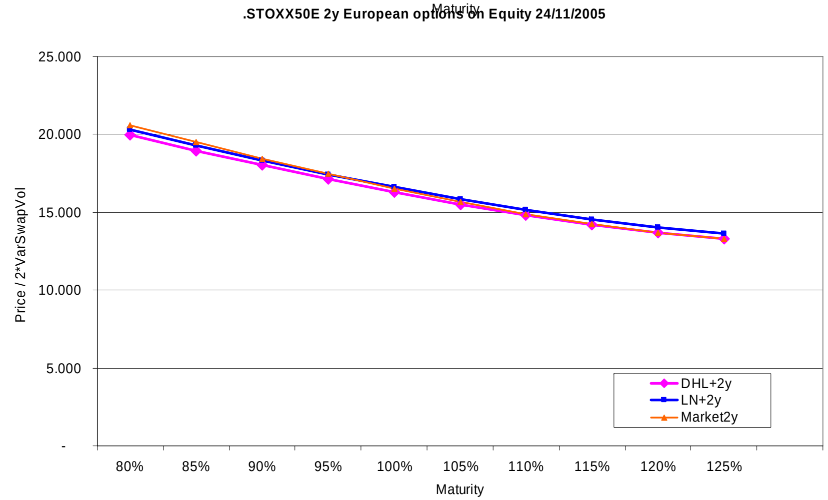
Fit the market

Fitting stochastic volatility models









Note that the calibration was performed only with the variance swaps and the ATM equity options …we get a remarkably good fit to the market.

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Variance Curves

Future

Variance Curves

Future

- “Statistical” variance curve models: PCA of historic data
  - Work in progress (Kai Detlefsen, HU Berlin)

- Challenges ahead
  - Is it possible to go from the variance curve model to a stochastic implied volatility model - can the correlation function ρ be deducted from market data?
  - Incorporation of stochastic interest rates and dividends (in particular long-term deals could exhibit strong exposure to stochastic interest rates).
  - Jumps both in the underlying and the variance process (witness S&P return graph earlier).
  - Correlation between the variance curves between different underlyings.

Thank you very much for your attention.

Details on the material presented here can be found in “Consistent Variance Curve Models”, Finance and Stochastics (2006), and in the forthcoming “Equity Hybrid Derivatives” (2006).
Hedging in Markovian markets is to be discussed in “Hedging in Factor Models” (joint work with J.Teichmann).

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NB we are generally interested in internship projects !!

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