Consistent Variance Curve Models
Theory and Application

Imperial College, London
March 23rd, 2006

Hans Buehler
http://www.math.tu-berlin.de/~buehler/
hans.buehler@db.com

Deutsche Bank

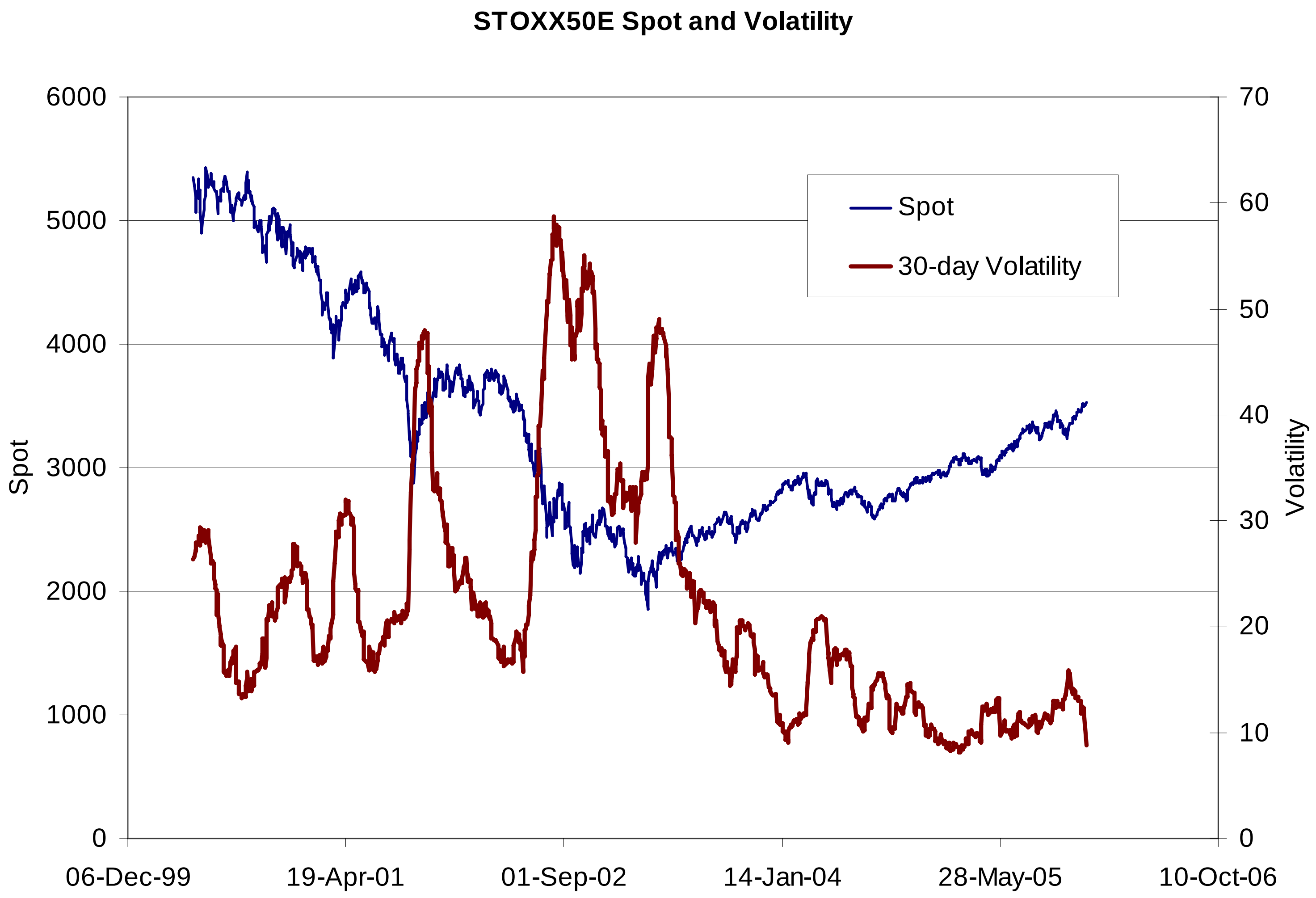
Consistent Variance Curve Models
Outline

- Introduction to Realized Variance
- Variance Curve Models
- Market Completeness
- Examples
- Outlook and comments to applications

Realized Variance

Trading volatility

Realized Variance
Introduction



Realized Variance

Introduction

- Equity market investors are interested in “trading volatility”
  - Speculation
  - Hedging
    - Ad-hoc “vega-hedging” against moves in volatility if Black&Scholes-type pricing models are used

- Traditionally, both have been implemented using European options.
- But European options are not very sensitive to volatility once spot moves away from the strike.
  - Why don’t we trade volatility directly?

Realized Variance

Introduction

- The *realized variance* of a stock price process $S=(S_t)_t$ over business days $0=t_0<\cdots<t_n=T$ is given as the unbiased estimator

$$\frac{252}{n}\sum_{i=1}^{n}\left(\log\frac{S_{t_t}}{S_{t_{t-1}}}\right)^2$$

  – Inherent “zero-mean” assumption.

  – For single stocks, dividends are taken out.

  – The $252/n \approx 1/T$ factor “annualizes” the variance.

Realized Variance

Introduction

- The previous definition also makes sense from a "stochastic analysis" viewpoint: if T is fixed but $n \uparrow \infty$, then we see that

$$\langle \log S \rangle_T \approx \sum_{i=1}^{n} \left(\log \frac{S_{t_t}}{S_{t_{t-1}}} \right)^2$$

  by definition of the quadratic variation.

  – This is also true if S has a drift and potentially jumps, hence the zero-mean assumption is justified in the limit.

- In the forthcoming discussion, we will assume that realized variance is defined as quadratic variation.

  – The error is discussed in Barndorff-Nielsen et al (2004).

Realized Variance
Assumptions

- Assume that S is continuous, that it pays no dividends and that the interest rates are zero. Hence*, we may write it on a stochastic base $(\Omega, \mathrm{P}, \mathrm{F})$ as

$$\begin{aligned} S_t &= \exp\left(X_t - \tfrac{1}{2}\langle X \rangle_t\right) \\ dX_t &= \sqrt{\zeta_t}\, dB_t \end{aligned}$$

  - The one-dimensional Brownian motion B is adapted to the filtration F.
  - The *short variance* process ζ is a predictable, integrable and non-negative.
  - Deterministic rates and proportional dividends can be taken into account (forthcoming “Equity Hybrid Derivatives”, 2006)

- Realized variance is then the non-negative quantity

$$\langle \log S \rangle_T = \int_0^T \zeta_s \, ds$$

* And that its quadratic variation is absolutely continuous wrt the Lebesgue measure.

Deutsche Bank

Realized Variance

Variance Swaps

- The simplest product on realized variance is a *variance swap*, which is essentially a forward on realized variance:
  - At maturity T it pays the variance realized during the life of the contract.
  - If an equivalent martingale measure exists, then the price $V_t(T)$ of a zero strike variance swap is therefore just the expectation of the realized variance,

$$V_t(T) := \mathrm{E}\left[\int_0^T \zeta_s \, ds \mid \mathrm{F}_t\right]$$

  - In practise, a variance swap pays the annualized realized variance in exchange for a fixed strike K.

$$\tfrac{1}{T} V_t(T) - K^2$$

But we ignore this deterministic transformation here for ease of notation.

Realized Variance

Variance Swaps

- If European options are traded for all strikes, the price of a variance swap can in theory be computed in terms of European options using Neuberger's (1990) formula,

$$
\begin{aligned}
V_0(T) &= 2\,\mathrm{E}\left[-\int_0^T \sqrt{\zeta_s}\,dB_s + \tfrac{1}{2}\int_0^T \zeta_s\,ds\right] \\
&= 2\,\mathrm{E}\left[S_T - 1 - \log S_T\right] \\
&= 2\left\{\int_0^1 \frac{1}{K^2}\mathrm{Put}(T,K)\,dK + \int_1^\infty \frac{1}{K^2}\mathrm{Call}(T,K)\,dK\right\}
\end{aligned}
$$

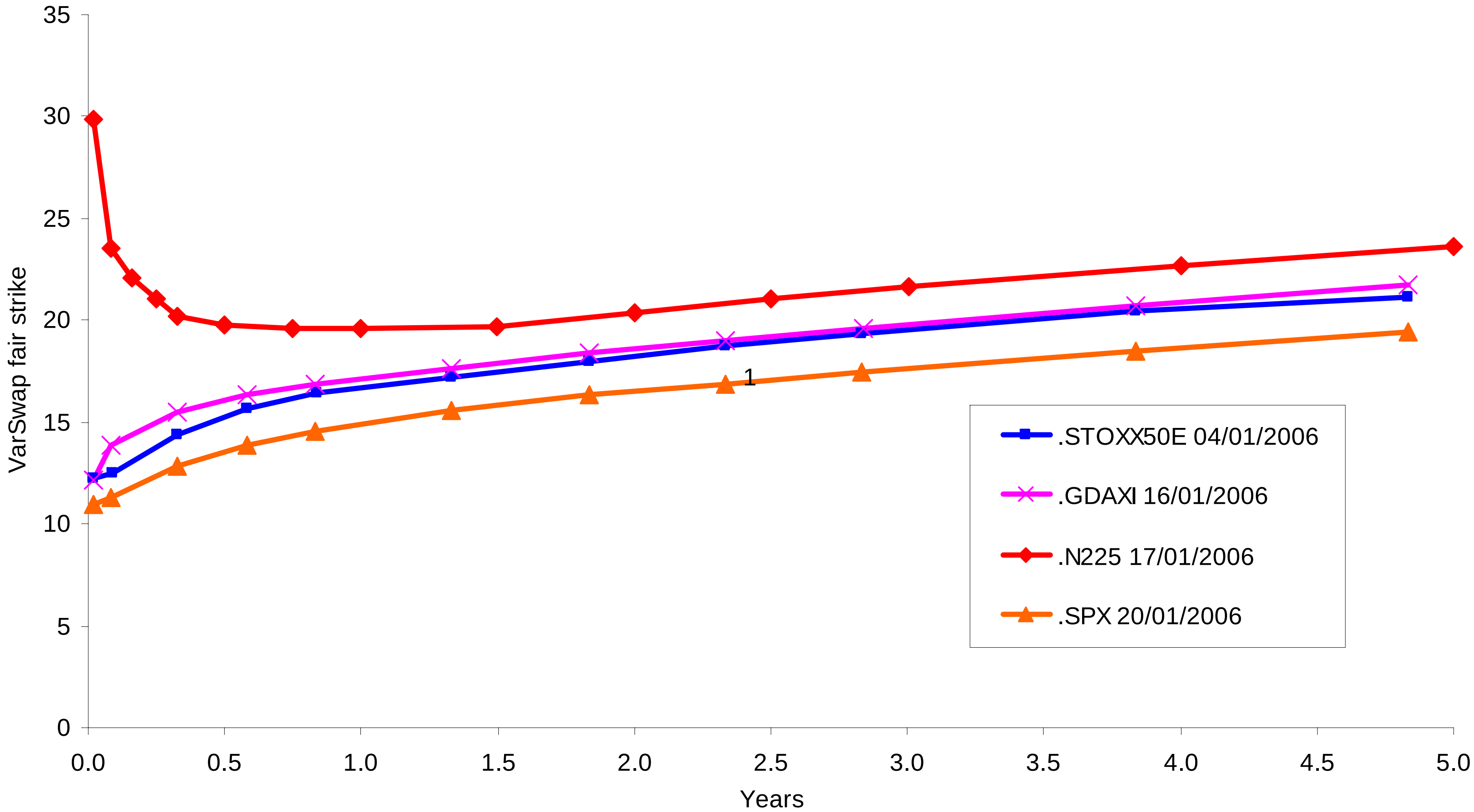
- The formula probably contributes to the fact that variance swaps are now liquidly traded for all major indices.
  - An excellent reference is Demeterfi et al (1999).

Realized Variance

Variance Swaps

Prices are quoted in "volatility" $\sqrt{\frac{1}{T}\int_0^T \zeta_s ds}$

Variance Swap Market Prices



Deutsche Bank

Realized Variance

Variance Swap Markets

- In particular in the US, the variance swap market is very liquid.
  - Spread in terms of volatility is just 0.4 vol points, compared with 0.2 vol points for ATM European options.
  - Bloomberg started quoting variance swaps Jan 06 – until now OTC.

- Since variance swaps are liquidly traded, there is no need to price and hedge them …

Realized Variance

Beyond Variance Swaps

… but what about more complex products:

– *Calls* on realized variance

$$\left(\int_0^T \zeta_s ds - K^2\right)^+$$

– *Volatility swaps*

$$\sqrt{\int_0^T \zeta_s ds} - K$$

– But also *forward started options* on the stock

$$\left(S_{T_2} \big/ S_{T_1} - k\right)^+ = \left(\exp\left(\int_{T_1}^{T_2} \sqrt{\zeta_s}\, dB_s - \tfrac{1}{2}\int_{T_1}^{T_2} \zeta_s ds\right) - k\right)^+$$

Realized Variance

Modelling volatility

- For such products, it should intuitively be sufficient to model only the variance swaps: the idea is that variance swaps can be used to "delta-hedge" more complex options on realized variance.
  - Of course, to obtain a useful model, we will also want to model the stock price itself and to develop a good concept of "skew".

- Mathematically, the term-structure of variance swaps reminds on the term-structure of discount bounds in interest rate models.
  - It is therefore tempting to apply concepts from interest rate theory to the pricing of options on variance.

Variance Curve Models

Modelling volatility

Variance Curve Models

Program

- Instead of starting with S as in classic stochastic volatility models, let us first specify the dynamics of the variance swaps .
- Then, construct a (local) martingale S which has the correct quadratic variation such that the market of variance swaps and stock is free of arbitrage.
- The correlation between the Brownian motion which drives S and the variance curve will act as a skew parameter.
- Since we are fundamentally aiming at replication, we provide criteria when the market is complete.
  - This excludes per se jumps in our discussion.

Variance Curve Models

Forward Variance

- Variance swap prices are increasing with maturity T.
- Their price at a later time t also depends on the past realized variance.
- To alleviate these unpleasant properties, note that

$$V_t(T) = \mathrm{E}\left[\int_0^T \zeta_s ds \mid \mathrm{F}_t\right] = \int_0^T \mathrm{E}\left[\zeta_s \mid \mathrm{F}_t\right] ds$$

can be differentiated in T to define the *forward variance*

$$v_t(T) := \partial_T V_t(T) = \mathrm{E}\left[\zeta_T \mid \mathrm{F}_t\right]$$

Observation time **Maturity**

– Note the similarity to the *forward rate* in interest rate theory.

– An important property is that forward variance can be zero.

Variance Curve Models

Classic approach

- Assume we have a driving d-dimensional extremal Brownian motion W on the space $(\Omega,\mathrm{P},\mathrm{F})$.
- <u>Definition</u>
  A family $v=(v(T))_{T\geq 0}$ is called a [local] *Variance Curve Model* if

  1. For each $T>0$, the process $v(T)=(v_t(T))_{t\in[0,T]}$ is a non-negative [local] martingale:

$$dv_t(T) = \sum_{j=1,\ldots,d} \beta_t^j(T)\,dW_t^j \qquad \beta^j(T) \in L^{\mathrm{loc}}$$

Set of intregrable, predictable processes wrt W.

  2. For each $T>0$, the initial variance swap prices are finite, i.e.

$$V_0(T) = \int_0^T v_0(s)\,ds < \infty$$

  3. The curve $v_t(t)$ is left-continuous.

Deutsche Bank

Variance Curve Models

Classic approach

- <u>Properties</u>
  - The price processes of variance swaps,

$$V_t(T) := \int_0^T v_t(s)\,ds$$

are [local] martingales.

  - The *short variance process*

$$\zeta_t := v_t(t)$$

is well defined, integrable and non-negative.

Variance Curve Models

Classic approach

- Properties
  Given any standard Brownian motion B on $(\Omega, \mathrm{P}, \mathrm{F})$, the process

$$dX_t = \sqrt{\zeta_t}\, dB_t$$

  is a square-integrable martingale, so the via B *associated stock price*

$$S_t := \exp\left(X_t - \tfrac{1}{2}\langle X \rangle_t\right)$$

  is a local martingale.

  – B represents the *correlation structure* of S with v.

- Theorem
  For each variance curve model v and each Brownian motion B, the market

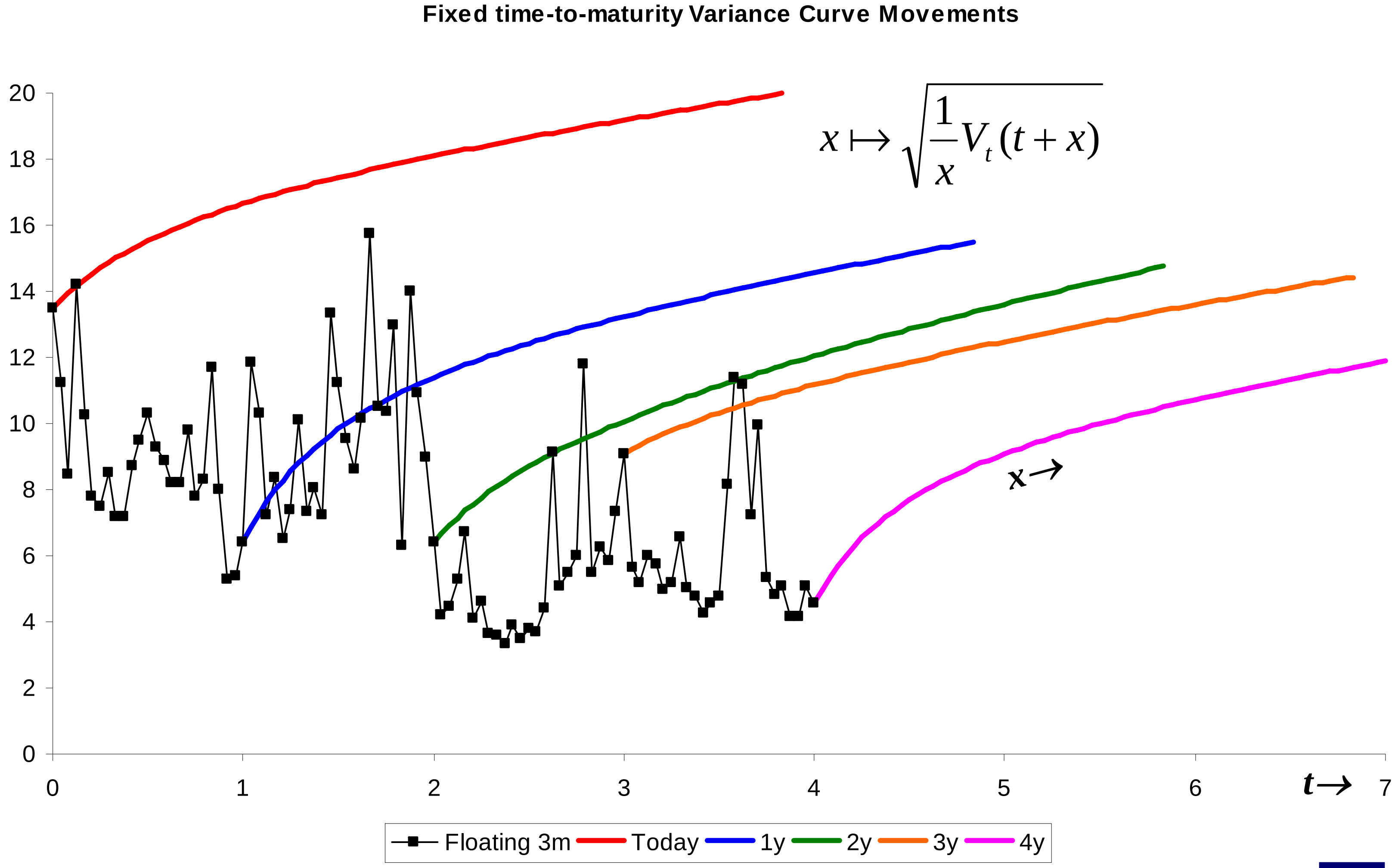
$$\left(S; (V(T))_{T \geq 0}\right)$$

  is free of arbitrage.

Deutsche Bank

Variance Curve Models

Classic approach –Musiela parametrization



Variance Curve Models

Classic approach – Musiela-Parametrization

- As in interest rates, it is more convenient to work with fixed time-to-maturities $x:=T\text{-}t$. Hence we define the *Musiela parameterization*

$$u_t(x) := v_t(t+x) \qquad\qquad v_t(T) = u_t(T-t)$$

- Starting in Musiela-parametrization
  - Assume that $\sum_{j=1,\ldots,d}\int_0^\infty\int_t^\infty \partial_T\beta_t(T)^2\,dT\,dt < \infty$
    Then,

$$du_t(x) := \partial_x u_t(x)dt + \sum_{j=1,\ldots,d} b_t^j(x)dW_t^j$$

defines a local variance curve model in Musiela-parametrization.

Deutsche Bank

Variance Curve Models

Classic approach – step one

- The previous discussion shows that it is remarkably easy to construct an arbitrage-free market with Variance Curve Models.
- Problems are
  1. The general predictable integrands are far too general (and it is difficult to check whether v remains non-negative).
  2. We would much prefer a representation in terms of a "driving" *finite-dimensional Markov process* to actually be able to implement the model on a computer.
  3. Finally, how do I fit an initial term structure from the market perfectly (cf. HJM models) ?

Variance Curve Models

Variance Curve Functionals – Finite Dimensional Parametrization

- Problems with a specification with general integrands $b(T)$:
  - It is complicated to check whether u remains non-negative.
  - In practice, it is not clear how to handle such integrands computationally.
- Ideally, we want to write

$$u_t(x) := G(Z_t; x)$$

  for some suitable non-negative function G and an m-dimensional Markov-process Z.
  - The function G is the “interpolation function” for the forward variances.

Variance Curve Models

Variance Curve Functionals

- Definition

1. A non-negative $C^{2,2}$-function $G:DxR^+ \to R^+$ is called a *Variance Curve Functional* if

$$\int_0^T G(z;x)dx < \infty$$

for all T and $z \in D$ where D is an open set in $R_{\geq 0}{}^m$.

2. We denote by Ξ the set of all $C=(\mu,\sigma)$ for which the SDE

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\dots,d} \sigma^j(Z_t)dW_t^j$$

starting at any point $Z_0 \in D$ has a unique solution Z which stays in D.

– Time-dependency is included in this setup.

Variance Curve Models

Variance Curve Functionals

- Definition
  We call $C=(\mu,\sigma)\in\Xi$ a *consistent factor model* for G if for any $Z_0 \in D$,

$$u_t(x) := G(Z_t; x)$$

  defines a local variance curve model.

- Theorem
  This is the case if and only if Z stays in D and if

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^T\sigma(z)\,\partial_{zz}^2 G(z;x)$$

  holds.

Variance Curve Models

Variance Curve Functionals

- Local Correlation and the Markov property
  Given a consistent factor model $C=(\mu,\sigma)\in\Xi$ and a "correlation function" $\rho: R^+ x D \to [-1,1]^d$ with $|\rho|=1$, we can always define

$$dS_t = \sum_{j=1,\dots,d} S_t \rho^j(S_t;Z_t)\left\{\sqrt{G(Z_t;0)}\,dW_t^j\right\}$$

  such that the process (S,Z) is Markov and S is a local martingale (note that the SDE does not explode).

  – The Markov property is essential for market completeness as we will see later.

- Local Volatility
  Local-Stochastic volatility "mixture models" are also part of this framework: they correspond to the case where S is one of the factors of Z.

Variance Curve Models

Term-structure approach

- The next logical step is to model the entire curve u as a process with values in a Hilbert space H.
  - We follow the path laid by Bjoerk/Christensen (1999), Filipovic (2000), Filipovic/Teichmann (2004) and Teichmann (2005).

- The main difference between variance curves and forward curves is that the curves u must remain non-negative (but *can* become zero).
  - The problem is that the “non-negative cone” is a very small set. Indeed it has no interior points.
  - However, if $G(D)$ is a sub-manifold with boundary of H, then it is sufficient to check whether u stays in $G(D)$.
    In this case we say $G(D)$ is *locally invariant* for u.
  - If G is moreover invertible, we can also directly construct a (locally) consistent factor model $C=(\mu,\sigma)$ for G.

Variance Curve Models

Term-structure approach

- Assume that the variance curve u is given as a solution in H to

$$du_t = \partial_x u_t dt + \sum_{j=1,\ldots,d} b^j(u_t) dW_t^j$$

where the coefficients β are locally Lipschitz vector fields.

- The Stratonovic-drift for u is as usual

Frechet-Derivative

$$\beta^0(u) := \partial_x u - \sum_{j=1,\ldots,d} D\beta^j(u) \cdot \beta^j(u)$$

such that

$$du_t = \hat{\beta}^0(u_t)dt + \sum_{j=1,\ldots,d} \hat{\beta}^j(u_t) \circ dW_t^j$$

Stratonovic-Integral

Deutsche Bank

Variance Curve Models

Term-structure approach

- Theorem (Filipovic/Teichmann 2004)
  The sub-manifold $G(D)$ is locally invariant for u iff

1. We have $G(D) \subset \mathrm{dom}(\partial x)$,
2. In the interior of $G(D)$, we have

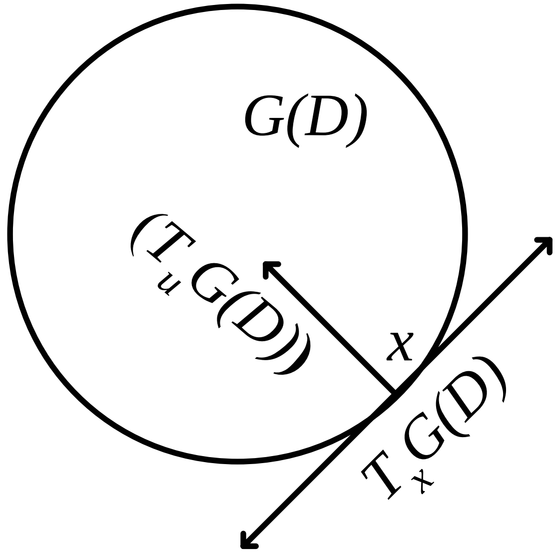
$$\hat{\beta}^j(u) \in T_u G(D) \qquad j = 0, \ldots, d$$

3. On the boundary $\partial G(D)$,

$$\hat{\beta}^0(u) \in (T_u G(D))_{\geq 0}$$

$$\hat{\beta}^j(u) \in T_u \partial G(D) \qquad j = 1, \ldots, d$$

holds.



Variance Curve Models

Term-structure approach

- If we can invert G, then $C=(\mu,\sigma)$ with

$$\sigma^j(z) := \partial_z G^{-1}(\hat{\beta}^j(G(z))$$

$$\mu(z) := \partial_z G^{-1}(\hat{\beta}^0(G(z)) + \sum_{j=1,..d} (\partial_z \sigma^j)(z)\sigma^j(z)$$

  is a consistent factor model for G.

- Hence, we have found a method to construct finite-dimensional parametrizations from Hilbert-space valued diffusions.

Hedging

Market Completeness

Hedging

General Markovian models

- Assume a market where the traded instruments $X=(X^1,\ldots,X^K)$ are given as a diffusion with volatilty matrix Σ.
- Market completeness is usually discussed in the context of all integrable, measurable non-negative payoffs.
  - In this case, standard theory says that the market is complete if Σ is non-singular (this allows recovery of the driving Brownian motion).
  - However, this is really stringent: it does not work for Heston, baskets of stocks with different holidays etc.
  - Anyway, why do we want to replicated payoffs whose value depend on the background BM?
    How can we determine the value of such a payoff?

- We present joint work with Josef Teichmann…

Hedging

General Markovian models

- <u>Definition:</u>
  Let $X=(X^1,\ldots,X^K)$ be tradable instruments. The market of *relevant payoffs* is the market of all L^1 random variables $H\geq 0$ which are measrurable wrt to X.

- <u>Theorem ("Delta hedging works"):</u>
  If $X=(X^1,\ldots,X^K)$ is a diffusion, and if the operator

$$f(t,x) := \mathrm{E}\left[F(X_t) \mid X_0 = x\right]$$

  maps C^∞ functions with compact support to C^1 functions, then the market of relevant payoffs is complete.
  We say that X "*weakly preserves smoothness*".

Hedging
General Markovian models

- Proposition:
  If $X=(X^1,\ldots,X^K)$ is a diffusion with differentiable matrix Σ with locally Lipschitz derivatives, then X *weakly preserves smoothness*.

- Ideas of proof (Buehler/Teichmann 2006)
  - Use local martingale property and Ito.
  - Approximiate non-smooth payoffs with Dirac sequences.
  - Generalize the usual way
  - The proposition is a consequence of the existence of a C^1 derivative of the process under the stated conditions.

- Handy tool to prove completeness.

Hedging

How to hedge with variance curve models

- We are back in our initial classical setting, i.e. we have decided to use a consistent variance curve model with a correlation function ρ such that

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\ldots,d} \sigma^j(Z_t)dW_t^j$$

$$u_t(x) = G(Z_t;x)$$

$$\zeta_t := u_t(0)$$

- In particular is (Z) Markov and the process can be checked to *weakly preserve smoothness* using our proposition. We assume it does.
- We want to hedge an option on variance,

$$H_T := h\left(\int_0^T \zeta_s ds\right)$$

Hedging

How to hedge with variance curve models

- Hence our candidate price process for h is the martingale:

$$H_t := \mathrm{E}\left[h\left(\int_t^T \zeta_s ds + V_t(t) \right) \mid \mathrm{F}_t \right] \qquad V_t(t) = \int_0^t \zeta_s ds$$

Running realized variance

- Due to the Markov-property of Z, we have

$$H_t = C_t(Z_t, V_t(t)) := \mathrm{E}\left[h\left(\int_0^T \zeta_s ds \right) \mid Z_t ; V_t(t) \right]$$

- The idea is now to express Z in terms of a finite number of variance swaps.

Hedging

How to hedge with variance curve models

- Define the *variance swap price* function

$$\overline{G}(z;x) := \int_0^x G(z;y)\,dy$$

and assume that there exist constant $0<\varepsilon<\tau_1<\cdots<\tau_m$ such that

$$\overline{G}_{t_1,\ldots,t_m}(z) := (\overline{G}(z;t_1),\ldots,\overline{G}(z;t_m))$$

is invertible for all $t_k := \tau_k - \tau$ for $0 \le \tau \le \varepsilon$.

- This then allows to recover Z in any interval $[a,b]$ by

$$Z_t = \overline{G}^{-1}_{T_1-t,\ldots,T_m-t}\left(V_t(T_1)-V_t(t),\ldots,V_t(T_m)-V_t(t)\right)$$

where $T_k := a + \tau_k$.

Variance swap

Running realized variance

Hedging

How to hedge with variance curve models

- Recall

$$H_t = C_t(Z_t, V_t(t)) := \mathrm{E}\left[h\left(\int_0^T \zeta_s ds \right) \Big| \; Z_t; V_t(t) \right]$$

- To hedge this payoff in the interval $[a,b]$, we can write it due to our assumptions on G as

$$C_t(Z_t, V_t(t)) \equiv C_t(V_t(T_1), \ldots, V_t(T_m), V_t(t))$$

Since G is C^2 and because Z weakly preserves smoothness, we get

$$dC_t(\cdots) = \sum_{k=1}^m \partial_{V_k} C_t(\cdots) dV_t(T_k)$$

Hedging
How to hedge with variance curve models

- This

$$dC_t(\cdots) = \sum_{k=1}^{m} \partial_{V_k} C_t(\cdots) dV_t(T_k)$$

  is the desired hedge of h in terms of variance swaps.
  - For options on variance, this is a “natural” hedge.
  - It can also be used for standard options (a delta-term for S will appear).
  - For forward started options, correlation (skew) risk should be taken into account.
- In practise, the above “VarSwapDelta” hedging ratios are computed via bumping of the variance swap price.

Back to reality

Applications

Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- <u>Example</u>
  A very basic example is the “linearly mean-reverting” functional:

„Speed of mean-reversion“

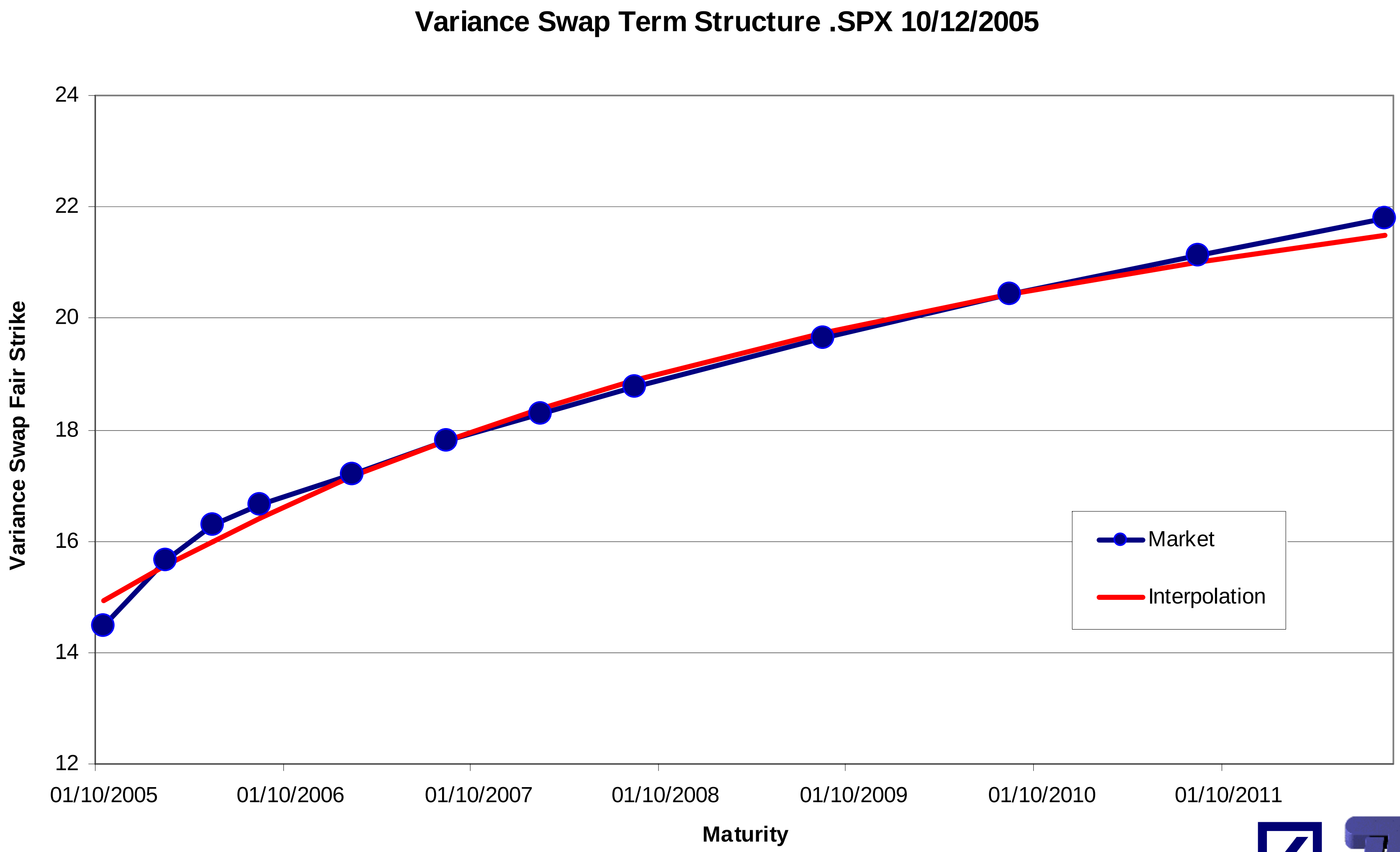
$$G(z;x) = z_2 + (z_1 - z_2)e^{-z_3 x}$$

„Long variance“ „Short variance“

for $z_1 \geq 0$ and $z_2, z_3 > 0$.

Variance Curve Models

Variance Curve Functionals – Linear mean-reversion



Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- Question: What dynamics can a consistent process $Z=(Z_1, Z_2, Z_3)$ have?
- The coefficients μ and σ have to satisfy

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^T\sigma(z)\partial_{zz}^2 G(z;x)$$

1. First, we see that $$\partial_{z_3 z_3}^2 G(z;x) = (z_1 - z_2)x^2 e^{-z_3 x}$$

   Since no term x^2e^x appears on the left hand side, we must have $\sigma_3=0$.
2. The same line of thought applied to

   $$\partial_{z_3} G(z,x) = -(z_1 - z_2)xe^{-z_3 x}$$

   shows that we also have $\mu_3=0$.

Hence, the speed of mean-reversion cannot be stochastic.

Variance Curve Models

Variance Curve Functionals – Linear mean-reversion

- For the other two parameters, we find that while σ is unconstrained,

$$\mu_2(z) = 0$$
$$\mu_1(z) = z_3(z_2 - z_1)$$

In other words: the only consistent processes for this choice of G are of Heston-type

Linear mean-reversion drift

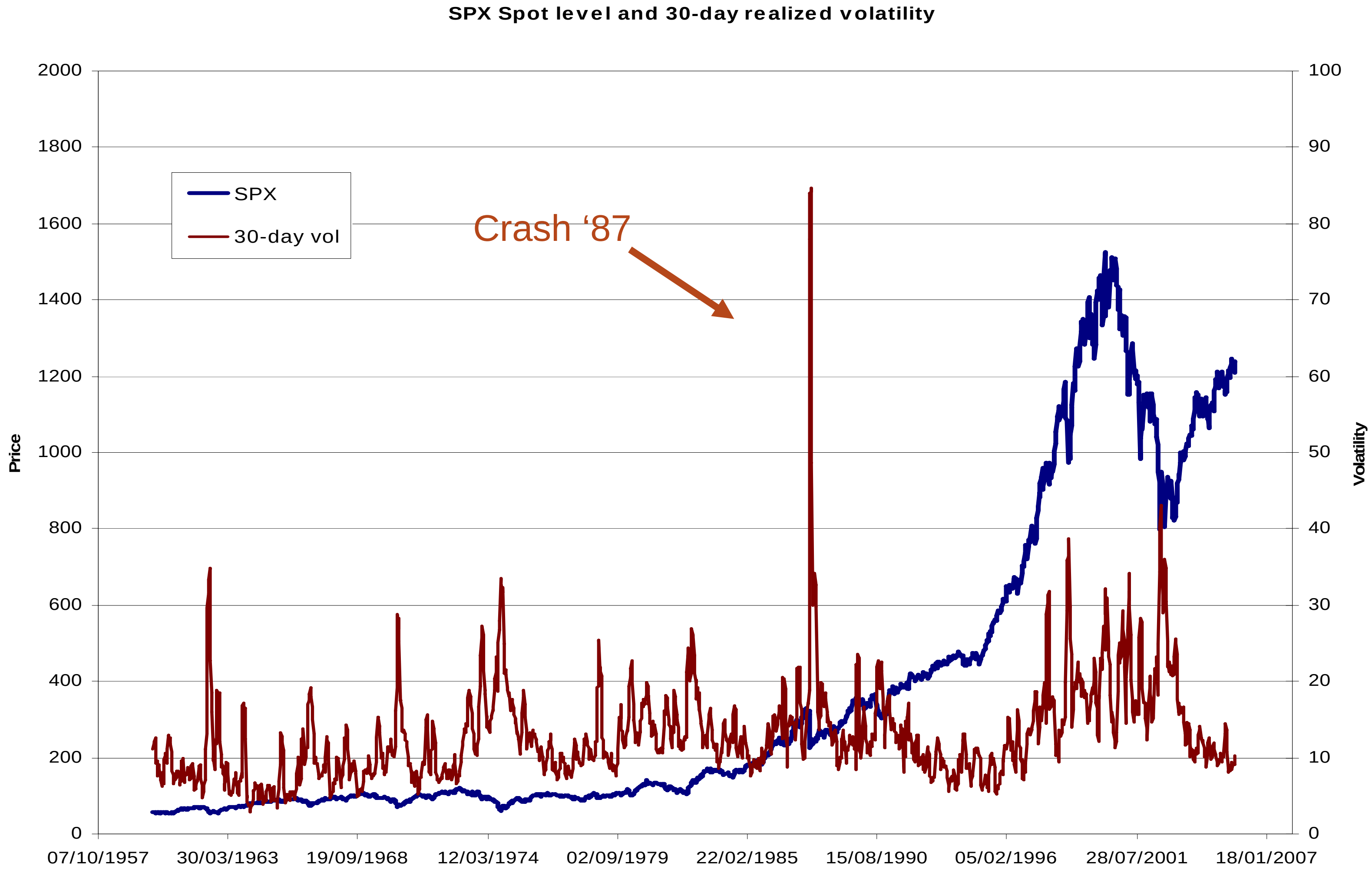
$$d\zeta_t = \kappa(\theta_t - \zeta_t)dt + \sigma_1(\zeta_t, \theta_t)dW_t$$
$$d\theta_t = \sigma_2(\zeta_t, \theta_t)dW_t$$

VolOfVol can freely be chosen as long as ζ remains non-negative.

Mean-reversion level θ is a positive martingale.

Variance Curve Models

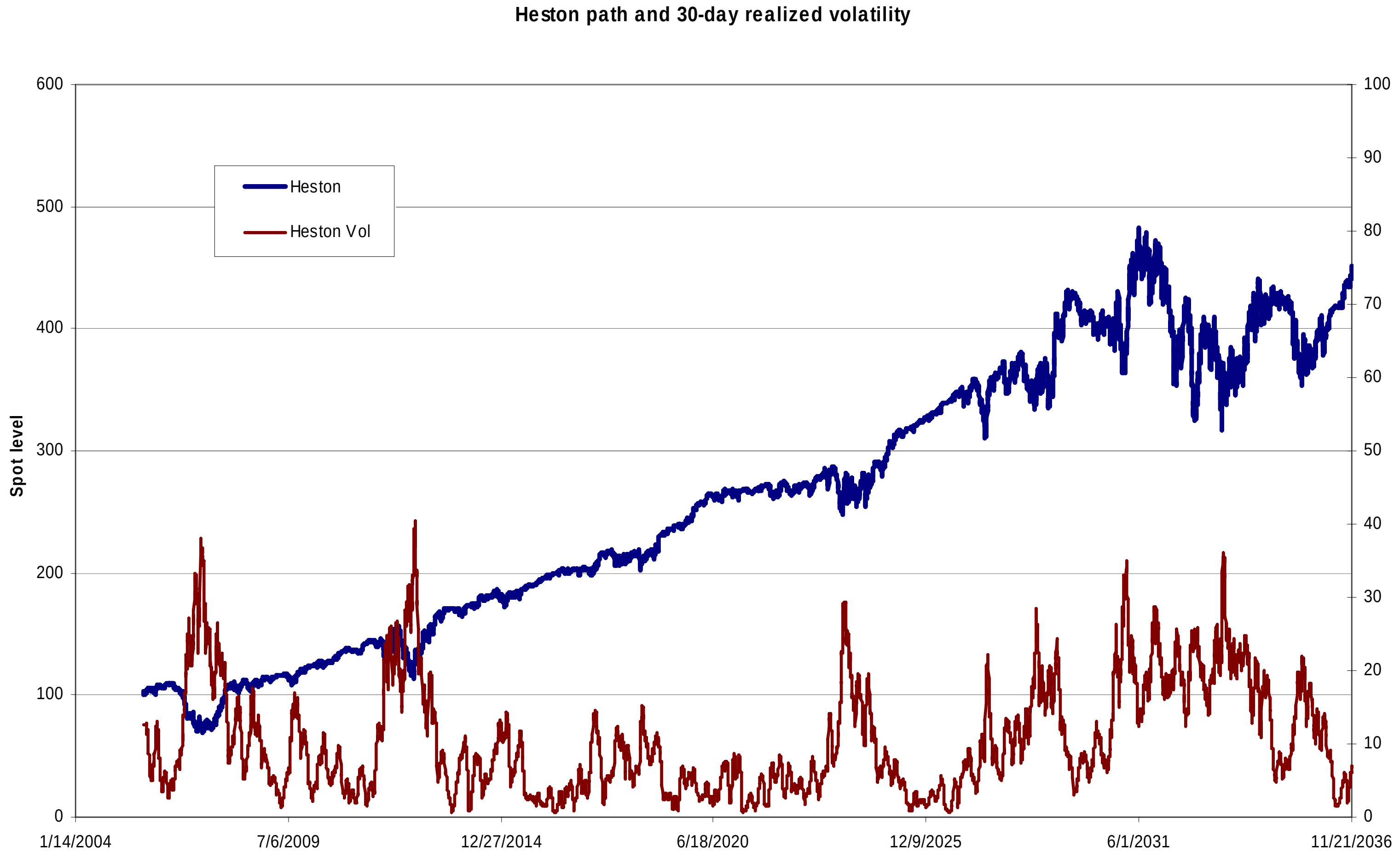
Why mean-reversion?



Variance Curve Models

Why mean-reversion?

Unconstrained Calibration	
ShortVol	14.4%
LongVol	28.7%
RevSpeed	0.23
Correlation	-0.74
VolOfVol	26.3%



Deutsche Bank

Variance Curve Models

Variance Curve Functionals

- <u>Proposition</u>
  The observation that mean-reversion speeds must be constant holds for all polynomial-exponential functionals, i.e. if $(p_i)_i$ are polynomials

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\sum_{i=1}^{n} p_i(z;x)e^{-z_i x}$$

  then the first n components must be constant (cf. Filipovic 2001 for interest rates).

- A similarly restrictive result can be shown for functionals of the form

$$G(z_1,\ldots,z_n,z_{n+1},\ldots,z_m;x)=\exp\left\{\sum_{i=1}^{n} p_i(z;x)e^{-z_i x}\right\}$$

  – The parameters in the exponent come in pairs, where one is twice as large as the other (again Filipovic 2001).

Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

- Another example of the polynomial-exponential class is

$$G(z;x) = z_3 + (z_1 - z_2)e^{-\kappa x} + (z_2 - z_3)\frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right)$$

  – A consistent factor model for this G must have the form

$$\begin{aligned} dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \sigma_1(Z_t)dW_t \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \sigma_2(Z_t)dW_t \\ dZ_t^3 &= \sigma_3(Z_t)dW_t \end{aligned}$$

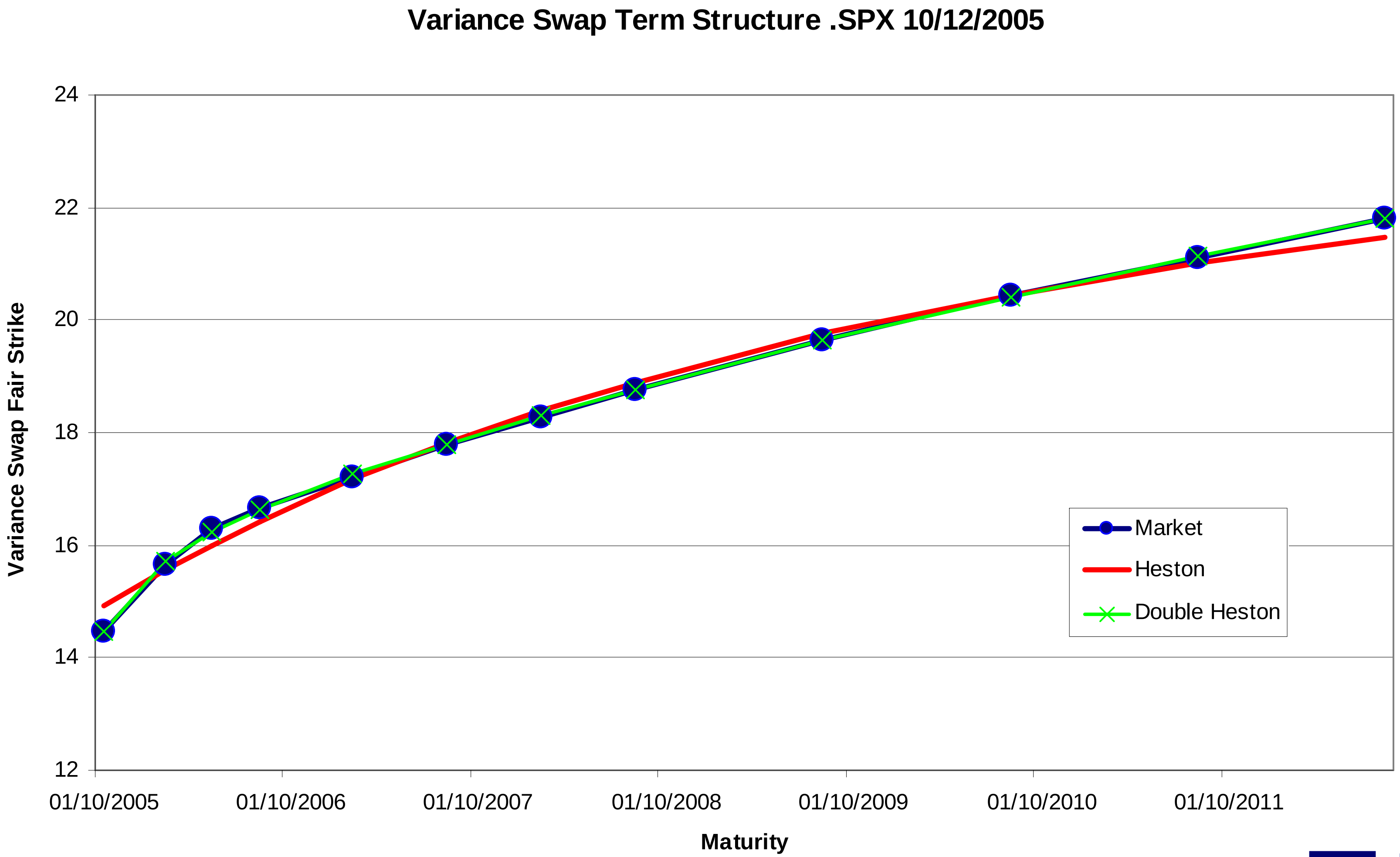
  which we call “double mean-reverting”.

  – Quite a good fit for most indices (at least during the course of the last year).

  – This is in effect Svensson’s interpolation function for interest rates.

Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion



Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion

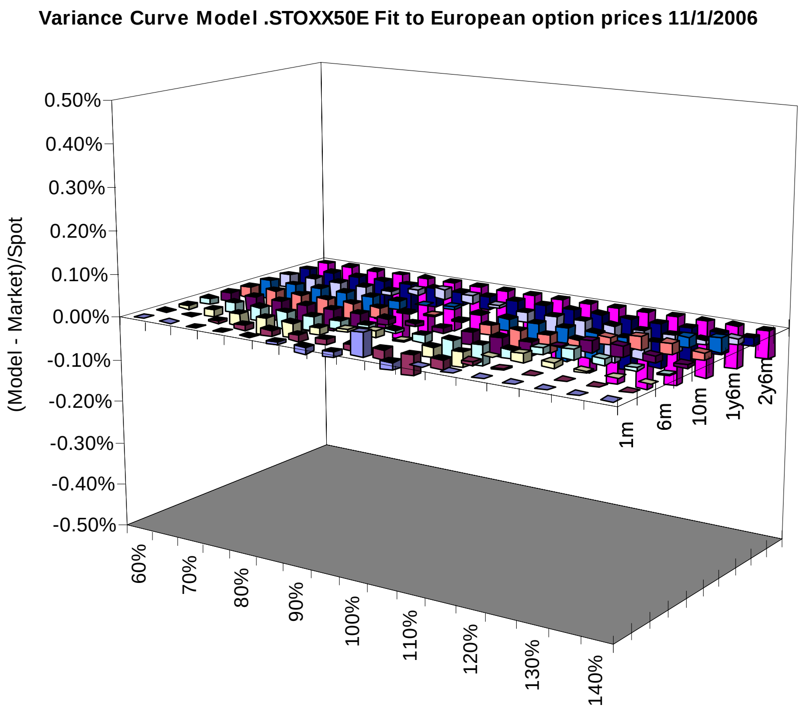
- Such a model is discussed in “Equity Hybrid Derivatives” (2006) where we used

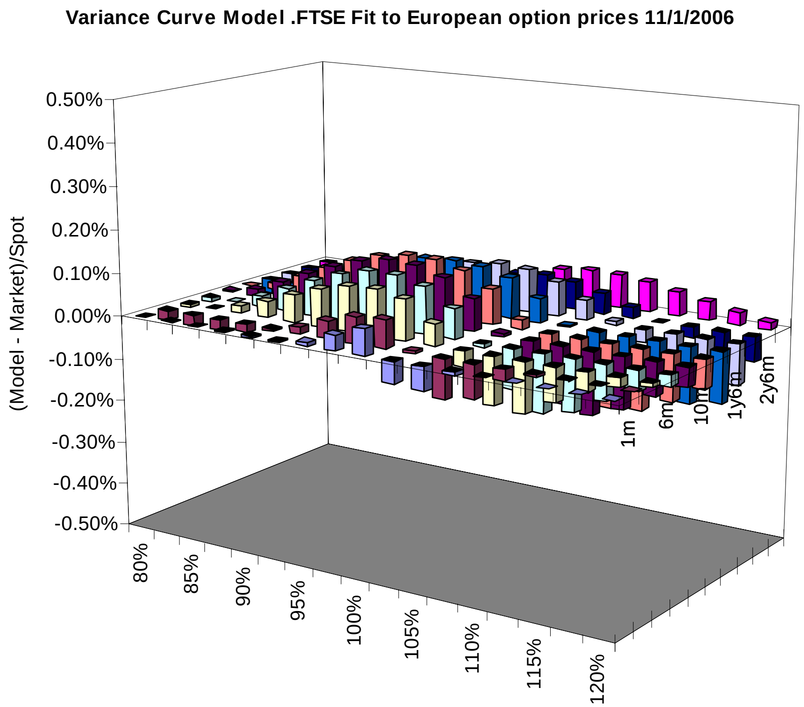
$$
\begin{aligned}
dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \nu(Z_t^1)^{\alpha}\, d\hat{W}_t^1 \\
dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \mu(Z_t^2)^{\beta}\, d\hat{W}_t^2 \\
dZ_t^3 &= \eta Z_t^3 d\hat{W}_t^3
\end{aligned}
$$

  for a correlated vector of Brownian motions.
  - To calibrate it, we first fit the variance curve function itself.
  - In a second step, we use European option prices close to ATM to calibrate the volatility and correlation parameters.
  - Numerically quite tedious.
- Intuitively, a more-factor model is only necessary if we want to price options on variance swaps etc.

Variance Curve Models

Variance Curve Functionals – Example linear mean-reversion





Examples of calibrating the three-factor model to real market data.

Deutsche Bank

Variance Curve Models

Classic approach – Fitting the market

- The idea of fitting a parametric function G to the market data is appealing from a "structural" point of view.
  - Clear vision of the variance swap curve and its movements.
  - Low-dimensional driving parameter prozess.
- Problem:
  - The variance swap market is too liquid to allow for mispricing of variance swaps.
  - Hence, a method of "fitting to the market" is sometimes required.
    (cf. interest rate theory)

Variance Curve Models

Classic approach – Fitting the market

- Assume that $(C=(\mu,\sigma), G)$ is a truly consistent pair and assume that we observe in the market a short variance curve $m_0(x)$.
  Then,

$$u_t(x) := \frac{m_0(t+x)}{G(Z_0;t+x)} G(Z_t;x)$$

"fits the market" in the sense that

$$\mathrm{E}\left[v_T(T)\right] = \mathrm{E}\left[u_T(0)\right] = m_0(T)$$

Examples of this approach (in principle)

– Dupire (2004): (C,G) is log-normal.

– Bergomi (2005): (C,G) is 2F exponential-OU

– Buehler (2005): (C,G) is Heston or the model we saw before

Variance Curves

Future

Variance Curves
Future

- "Statistical" variance curve models: PCA of historic data
  - Work in progress (Kai Detlefsen, HU Berlin)

- Challenges ahead
  - Is it possible to go from the variance curve model to a stochastic implied volatility model - can the correlation function ρ be deducted from market data?
  - Incorporation of stochastic interest rates and dividends (in particular long-term deals could exhibit strong exposure to stochastic interest rates).
  - Jumps both in the underlying and the variance process (witness S&P return graph earlier).
  - Correlation between the variance curves between different underlyings (our suspicion is that it is actually quite high).

Thank you very much for your attention.

Details on the material presented here can be found in “Consistent Variance Curve Models”, Finance and Stochastics (2006), and in the forthcoming “Equity Hybrid Derivatives” (2006).
Hedging in Markovian markets is to be discussed in “Hedging in Factor Models” (joint work with J.Teichmann).

http://www.math.tu-berlin.de/~buehler/
http://www.dbquant.com

hans.buehler@db.com
NB we are generally interested in internship projects !!

References

- O.Barndorff-Nielsen, S.Graversen, J.Jacod, M.Podolskij, N.Shephard: “A Central Limit Theorem for Realised Power and Bipower Variations of Continuous Semimartingales", WP 2004 (from their website)
- L.Bergomi: ”Smile Dynamics II", Risk September 2005
- T.Bjoerk, B. Christenssen: "Interest rate dynamics and consistent forward rate curves", Mathematical Finance, 9:4, 323-348, 1999.
- A.Brace, B.Goldys, F.Klebaner, R.Womersley: “A Market Model of Stochastic Implied Volatility with Application to the BGM model", WP 2001
- H.Buehler: “Consistent Variance Curve Models”, to appear in: Finance and Stochastics, 2006
- H.Buehler, J.Teichmann: “Hedging in Markov Models”, WP, 2005
- R.Cont, J.da Fonseca, V.Durrleman “Stochastic models of implied volatility surfaces", Economic Notes, Vol. 31, No 2, 361-377 (2002).
- B.Dupire: “Arbitrage Pricing with Stochastic Volatility", Derivatives Pricing, p.197-215, Risk 2004
- D.Filipovic, *Consistency Problems for Heath-Jarrow-Morton Interest Rate Models* (Lecture Notes in Mathematics 1760), Springer-Verlag, Berlin, 2001
- D.Filipovic, J.Teichmann: “Existence of invariant Manifolds for Stochastic Equations in infinite dimension”, Journal of Functional Analysis 197, 398-432, 2003.
- D.Filipovic, J.Teichmann: “On the Geometry of the Term structure of Interest Rates”, Proceedings of the Royal Society London A 460, 129-167, 2004.
- S. Heston: “A closed-form solution for options with stochastic volatility with applications to bond and currency options", Review of Financial Studies, 1993
- M.Overhaus, A. Bermudez, H.Buehler, A.Ferraris, C.Jordinson, A.Lamnouar: “Equity Hybrid Derivatives”, forthcoming, Wiley 2006
- A.Neuberger: “Volatility Trading", London Business School WP (1999)