Stochastic Dividend Modeling

For Derivatives Pricing and Risk Management

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J.P.Morgan
QUANTITATIVE RESEARCH

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- Part IV
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Presentation will be under http://www.math.tu-berlin.de/~buehler/

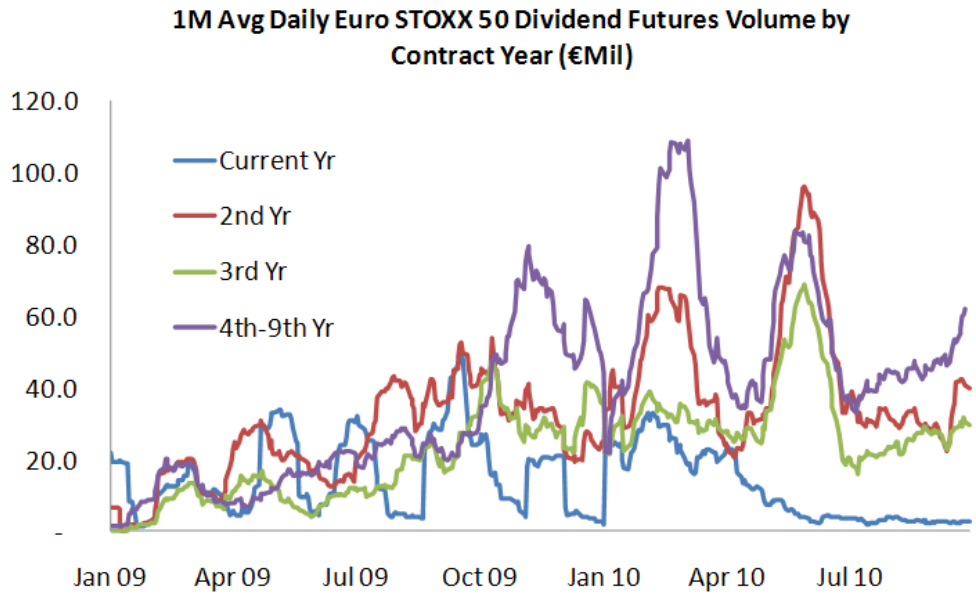
Part I
Vanilla Dividend Market

Dividend Futures

- Dividend future settles at the sum of dividends paid over a period T_1 to T_2 for all members of an index such as STOXX50E.

$$\sum_{i=T_1}^{T_2} \Delta^i$$

- Standard maturities settle in December, so we have Dec 13, Dec 14 etc trading.



Vanilla Dividend Market

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Vanilla Options

- Refers to dividends over a period T_1 to T_2. Listed options cover Dec X to Dec Y.
- Payoff straight forward –

$$\left(\sum_{i=T_1}^{T_2} \Delta^i - K\right)^+$$

  note that dividends are not accrued.
- Note in particular that a Dec 13 option does not overlap with a Dec 14 option ... makes the pricing problem somewhat easier than for example pricing options on variance.

Market

- Active OTC market in EMEA
- EUREX is pushing to establish a listed market for STOXX50E
- At the moment much less volume than in the OTC market

Calls

	Ticker	Strike	DM	Bid	Ask	Last	OInt	Volm	IVM	DB	GB
Dec 12 (623d); CSize 100.00; DEDZ2 123.20											
1)	DEDZ2	65				56.23s					
2)	DEDZ2	70				51.54s					
3)	DEDZ2	75				46.90s					
4)	DEDZ2	80				42.30s					
5)	DEDZ2	85				37.75s	1360				
6)	DEDZ2	90				33.23s					
7)	DEDZ2	95				28.74s	1360				
8)	DEDZ2	100				24.31s	1543				
9)	DEDZ2	105				19.96s	2500				
10)	DEDZ2	110				15.77s	2147				
11)	DEDZ2	115				11.88s	1080				
12)	DEDZ2	120				8.45s	1853				
13)	DEDZ2	125				5.60s	1250				
14)	DEDZ2	130				3.43s	14250				

	Ticker	Strike
Dec 12 (623d); C		
54)	DEDZ2	65
55)	DEDZ2	70
56)	DEDZ2	75
57)	DEDZ2	80
58)	DEDZ2	85
59)	DEDZ2	90
60)	DEDZ2	95
61)	DEDZ2	100
62)	DEDZ2	105
63)	DEDZ2	110
64)	DEDZ2	115
65)	DEDZ2	120
66)	DEDZ2	125
67)	DEDZ2	130

Quoting

- The first task at hand is now to provide a "Quoting" mechanism for options on dividends – this does not intend to *model* dividends; just to map market $ prices into a more general "implied volatility" measure.
- For our further discussion let t^* be $t^* := \max\{T_1, t\}$ and

$$\sum_{i=T_1}^{T_2} \Delta^i = \mathrm{Fut} + \mathrm{Past}$$

$$\mathrm{Fut} = \sum_{i=t^*}^{T_2} \Delta^i, \ \mathrm{Past} = \sum_{i=T_1}^{t^*} \Delta^i, \ \mathrm{EFut} := \mathrm{E}_t[\mathrm{Fut}]$$

- The simplest quoting method is as usual Black & Scholes:

Imply volatility from the market.

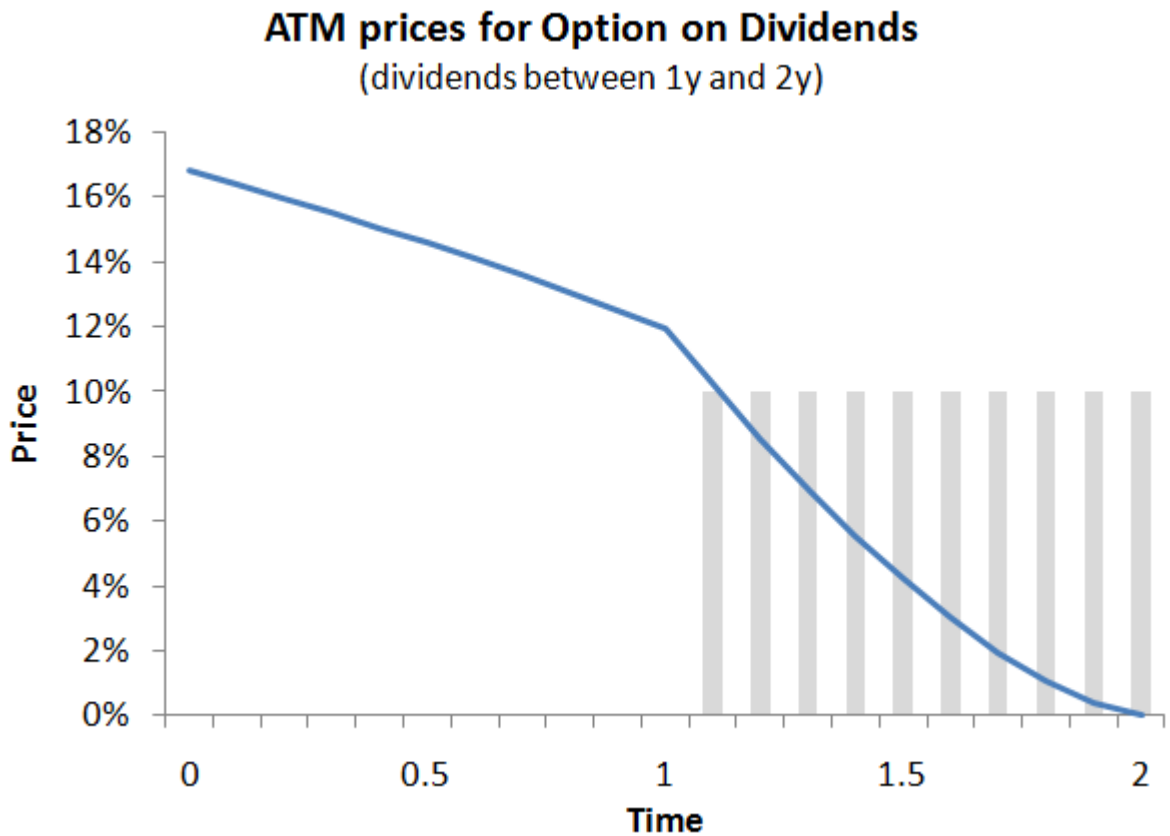
$$\mathrm{E}_t\left(\sum_{i=T_1}^{T_2} \Delta^i - K\right) := \mathrm{BS}\left(\mathrm{EFut}, K - \mathrm{Past}; T_2 - t, \sigma_{div}\right)$$

BS forward equal to expected future dividends

Adjust strike by past dividends.

Quoting

... term structure looks a bit funny though.



Graph shows ATM prices for option son div for the period $T_1=1$ and $T_2=2$ at various valuation times.

Quoting

- Basic issue is that dividends are an "average" so using straight Black & Scholes doesn't get the decay right.
- Alternative is to use an average option pricer – for simplicity, use the classic approximation

$$\frac{1}{x}\sum_{i=0}^{x}\Delta^i = \frac{1}{x}\int_0^x e^{\sigma W_s ds+\ldots} = e^{\frac{1}{x}\int_0^x \sigma W_s ds+\ldots} \cong e^{\sigma\sqrt{\frac{x}{3}}Y+\ldots} = \sum_{i=0}^{x}\mathrm{E}[\Delta^i]\cdot e^{\sigma\sqrt{\frac{x}{3}}W_x-\frac{1}{6}\sigma^2 x}$$

and define the option price using BS' formula as

$$\mathrm{E}_t\left(\sum_{i=T_1}^{T_2}\Delta^i - K\right)$$

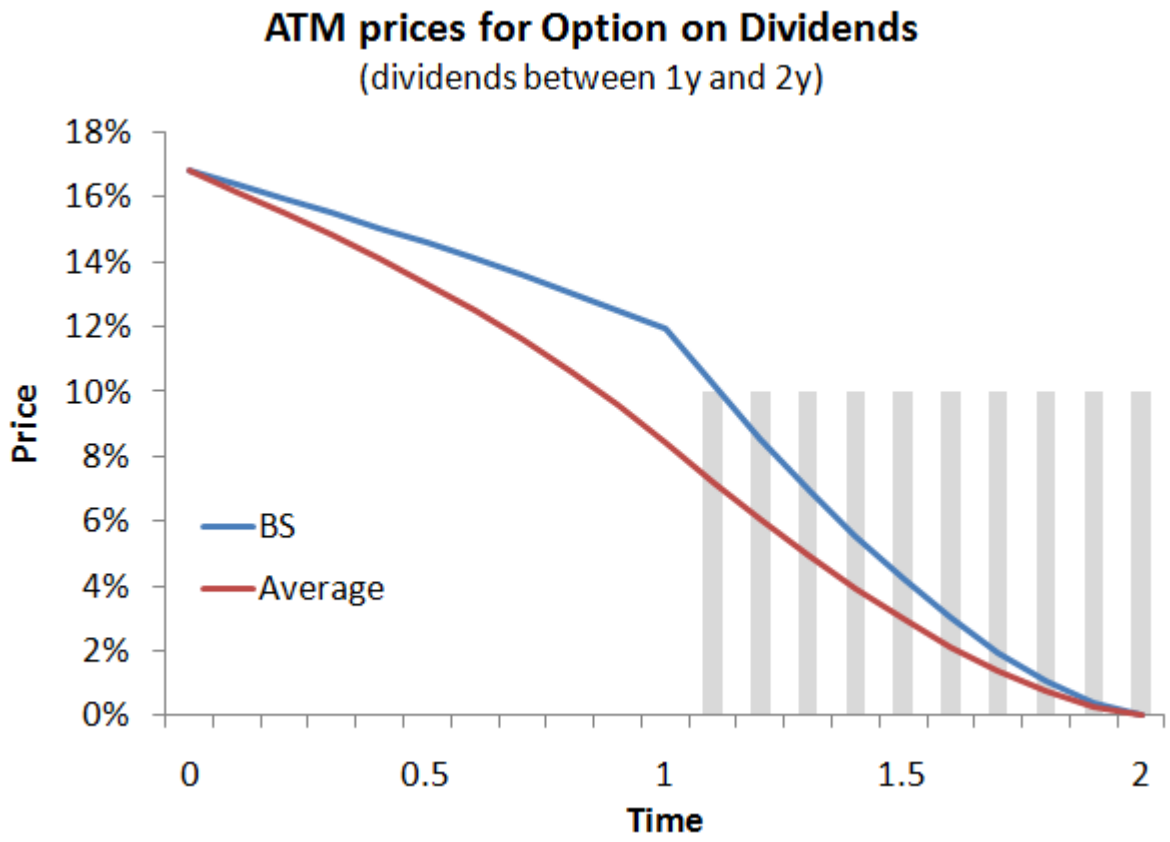
$$:= \mathrm{BS}\left(\mathrm{EFut}, K-\mathrm{Past}; (T_1-t_*)^+ + (T_2-t_*)/3, \hat{\sigma}_{div}\right)$$

Imply a different volatility from the market.

Basically the average pricing translates to a new scaling in time.

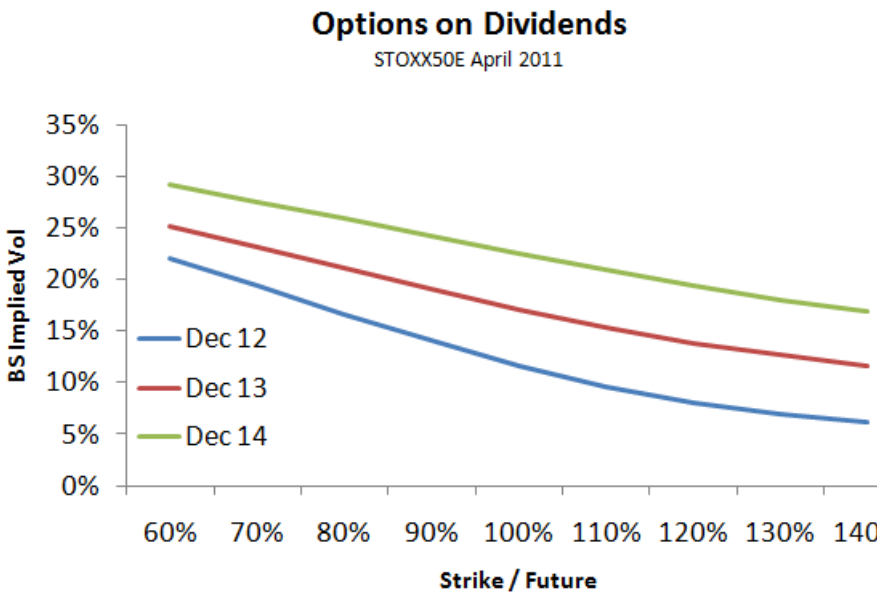
Quoting

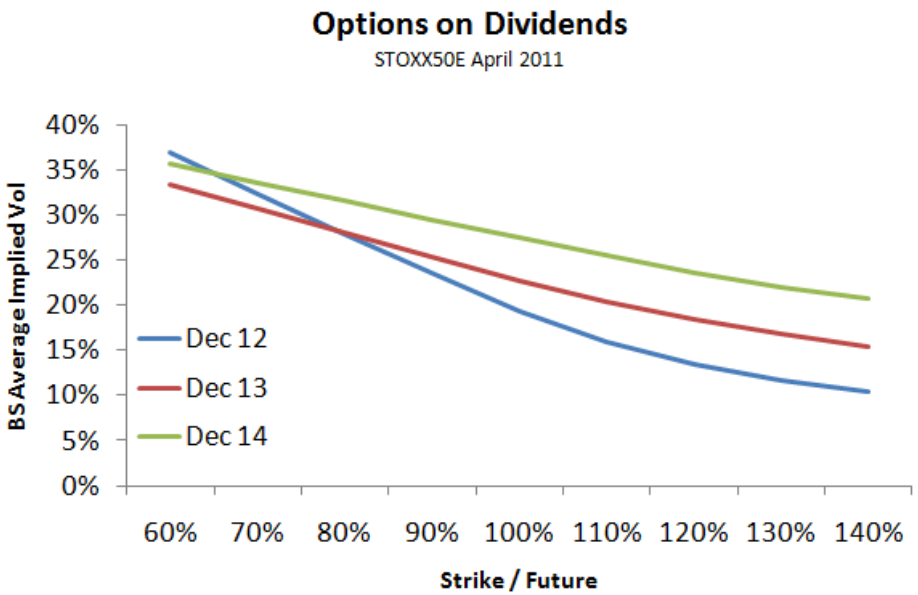
... gives much better theta:



Quoting

... market implied vols by strike:





Vanilla Dividend Market

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Quoting

- Using plain BS gives rise to questionably theta, in particular around T_1 → using an average approximation leads to much better results.
- After that, market quotes can be interpolated with any implied volatility "model".

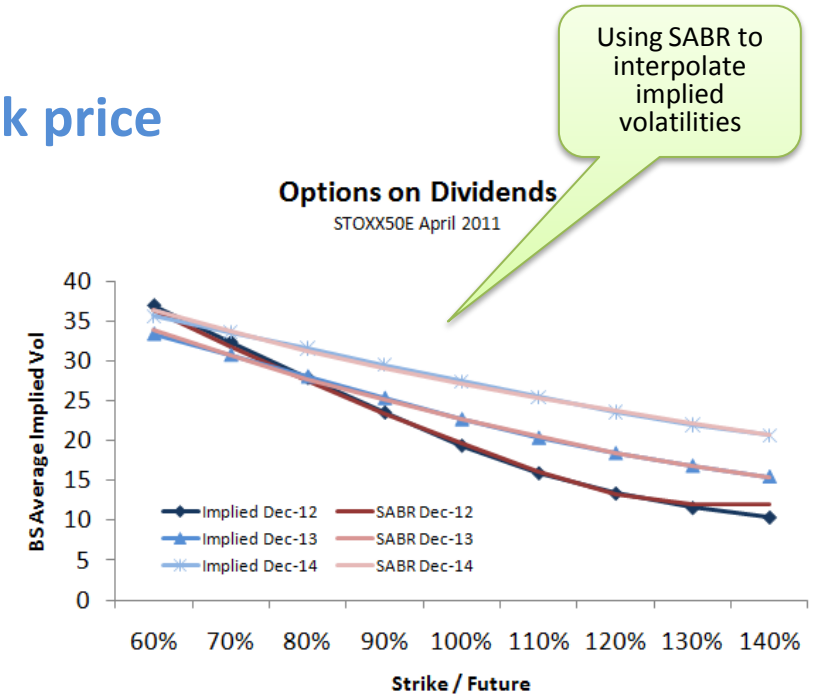
At that level no link to the actual stock price

→ let us focus on that now.

	Dec 12	Dec 13	Dec 14
α_0	25%	25%	31%
ρ	-0.85	-0.84	-9.59
ν	102%	47%	28%

$$d\Delta_t = \alpha_t \sqrt{\Delta_t}\, dW_t$$

$$d\alpha_t = \alpha_t \nu (\rho dW_t + \sqrt{1-\rho^2}\, dW_t^{\perp}$$



Part II
The Structure of Dividend Paying Stocks

The Structure of Dividend Paying Stocks

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Assumptions on Dividends

- We assume that the ex-div dates $0 < \tau_1 < \tau_2 < \ldots$ are known and that we can trade each dividend in the market.
- We further assume that dividends are $F_{\tau k-}$-measurable non-negative random variables.
  - We also assume that our dividends Δ^k are already adjusted of tax and any discounting to settlement date, i.e. we can look at the dividend amount at the ex-div date.

Other Assumptions

- We have an instantaneous stochastic interest rates r.
- The equity earns a continuous repo-rate μ.
- We assume $S_t > 0$ (it is straight forward to incorporate simple credit risk [1,2] but we'll skip that here)

Stock Price Dynamics

- In the absence of friction cost, the stock price under risk-neutral dynamics has to fall by the dividend amount in the sense that

$$\mathrm{E}_{\tau_k -}\left[S_{\tau_k}\right] = S_{\tau_k -} - \Delta^i$$

For example, we may consider an additional uncertainty risk in the stock price at the open:

$$S_{\tau_k} = (S_{\tau_k -} - \Delta^i) e^{wY_\kappa - \frac{1}{2}w^2}$$

- For the case where S has almost surely no jumps at τ_k we obtain the more common

$$S_{\tau_k} = S_{\tau_k -} - \Delta^k$$

Stock Price Dynamics

– In between dividend dates, the risk-neutral drift under any risk-neutral measure is given by rates and repo, i.e. we can write the stock price between dividend dates for t: $\tau_k < t \leq \tau_{k+1}$ as

$$S_t = S_{\tau_k} R_t^{\tau_k} Z_t^{(k)} \qquad R_t := e^{\int_0^t r_s - \mu_s ds} \qquad R_T^t := \frac{R_T}{R_t}$$

where $Z^{(k)}$ is a non-negative (local) martingale with unit expectation which starts in τ_k .

R can be understood as the "funding factor" of the equity

Stock Price Dynamics - Warning

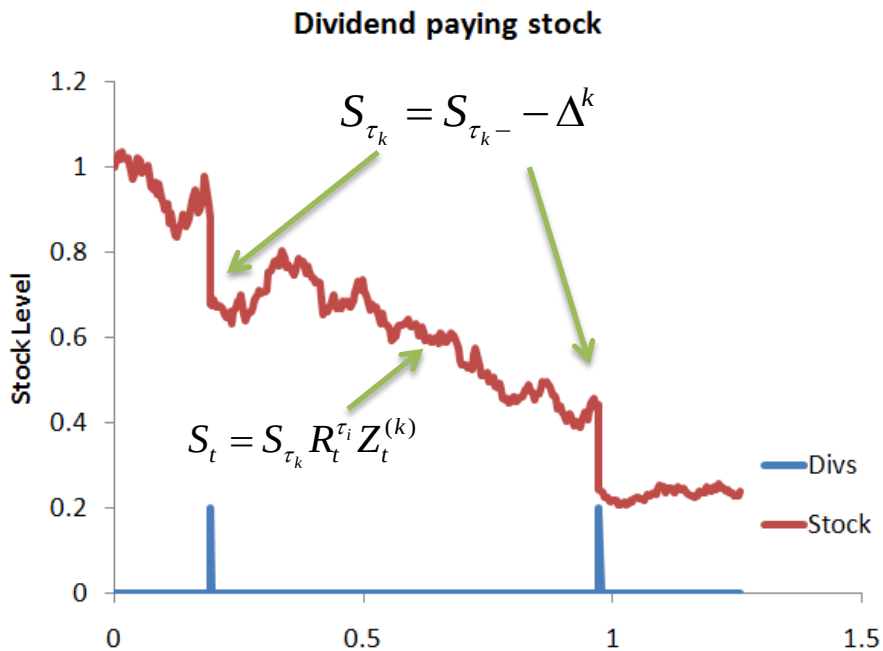
– This gives

Funding rate between dividends

(Local) martingale part between dividends

$$S_{\tau_k} = S_{\tau_{k-1}} R_t^{\tau_{k-1}} Z_t^{(k-1)} - \Delta^k$$

→ the martingale Z can *not* have arbitrary dynamics but needs to be floored to ensure that the stock price never falls below any future dividend amount.



Towards Stock Price Dynamics with Discrete Dividends

- In other words, any generic specification of the form

$$dS_t = S_t\left(r_t - \mu_t\right)dt + S_t \frac{dZ_t}{Z_t} - \sum_k \Delta^k \delta_{\tau_k}(dt)$$

  does not work either - a common fix in numerical approaches is to set $\Delta^k := \min\{S_{\tau_k -} + \varepsilon, \tilde{\Delta}^k)$

- Intuitively, the restriction is that the stock price at any time needs to be above the discounted value of all future dividends:
  - otherwise, go long stock and forward-sell all dividends → lock-in risk-free return.

Theorem (extension of Buehler 2007 [2])

- The stock price process remains positive if and only if it has the form

$$S_t := R_t\left(\tilde{S}_0 Z_t + \sum_{k:\tau_k > t} D_t^k\right)$$

Ex-dividend stock price

Discounted value of all future dividends

- where the positive local martingale Z is called the pure martingale of the stock price process.

$$\tilde{S}_0 := S_0 - \sum_{k:\tau_k > 0} D_0^k \qquad D_t^k := \mathrm{E}_t\left[R_0^{\tau_k}\Delta^k\right]$$

- The extension over [2] is that this actually also holds in the presence of stochastic interest rates and for *any* dividend structure, not just affine dividends as in [2].

Consequences

– In the case of deterministic rates and borrow, we get [1], [2]:

$$S_t = (F_t - A_t)Z_t + A_t \qquad A_t := R_t \sum_{k:\tau_k > t} D_t^k$$

with forward

$$F_t = R_t S_0 Z_t + R_t \sum_{k:\tau_k \le t} D_t^k$$

The stochasicity of the equity comes from the excess value of S over its future dividends.

This structure is not an assumption – it is a *consequence* of the assumption of positivity $S>0$!

All processes with discrete dividends look like this.

Structure of Dividends

- A consequence of the aforementioned is that we can write all dividend models as follows:
  - We decompose

$$\Delta^k := \Delta^k_{\min} + \tilde{\Delta}^k$$

$$\Delta^k_{\min} := \min \Delta^k \,,\; \tilde{\Delta}^k := \Delta^k - \Delta^k_{\min}$$

so that we can split effectively the stock price into a "fixed cash dividend" part and one where the dividends are stochastic:

$$S_t := R_t \left(\tilde{S}_0 Z_t + \sum_{k:\tau_k > t} \tilde{D}^k_t \right) + R_t \sum_{k:\tau_k > t} D^k_{t,\min}$$

Random dividends, floored at zero

Deterministic dividends

Exponential Representation Theorem

– Every positive stock price process $S>0$ which pays dividends Δ^k can be written in exponential form as

$$S_t := R_t\left(\tilde{S}_0 e^{X_t}\right) - A_{t,\min}$$

where A is given as before and where X is given in terms of a unit martingale Z as

$$X_t = \log(Z_t) - \sum_{k:\tau_k \le t} d_k(X_{t-})$$

with stochastic proportional dividends

$$d_k(X_{\tau_k -}) := -\log\left(1 - \frac{\tilde{\Delta}^k}{\tilde{S}_0 e^{X_{\tau_k -}}}\right)$$

Part III
Affine Dividends

Affine Dividends

- Black Scholes Merton: inherently supports proportional dividends.

$$\Delta^i := \beta_i S_{\tau_i -} \qquad (d_i \geq 0)$$

- Plenty of literature on general "affine" dividends, i.e.

$$\Delta^i := \alpha_i + \beta_i S_{\tau_i -}$$

All known approaches either:

- approximate by approach (i.e., the dividends are not affine).
- approximate by numerical methods

… but they fit well in out framework.

Structure of the Stock Price

$$\Delta^i := \alpha_i + \beta_i S_{\tau_i -}$$

- Direct application in our framework can be done but it is easier to simply write the proportional dividend effectively as part of the repo-rate μ (this is what happens in Merton 1973 [5]) i.e. write

$$R_t := e^{\int_0^t r_s - \mu_s ds} \prod_{i:\tau_i \le t} (1 - \beta_i)$$

- All previous results go through [1,2], i.e. we get

$$S_t = (F_t - A_t) Z_t + A_t \qquad A_t := R_t \sum_{k:\tau_k > t} D_t^k$$

with our new "funding factor" R.

Again – this structure is the *only* correct representation of a stock price which pays affine dividends.

Impact on Pricing Vanillas

– The formula

$$S_t = (F_t - A_t)Z_t + A_t$$

really means that we can model a stock price which pays affine dividends by modeling directly Z since:

$$\mathrm{E}(S_T - K)^+ = (F_T - A_T)\mathrm{E}\left(Z_T - \tilde{K}\right)^+ \qquad \tilde{K} := \frac{K - A_T}{F_T - A_T}$$

which means that we can easily compute option prices on S if we know how to compute option prices on Z.

Hence, Z can be any classic equity model

- Black-Scholes
- Heston, SABR, λ-SABR
- Levy/Affine
- Numerical Models (LVSV) ....

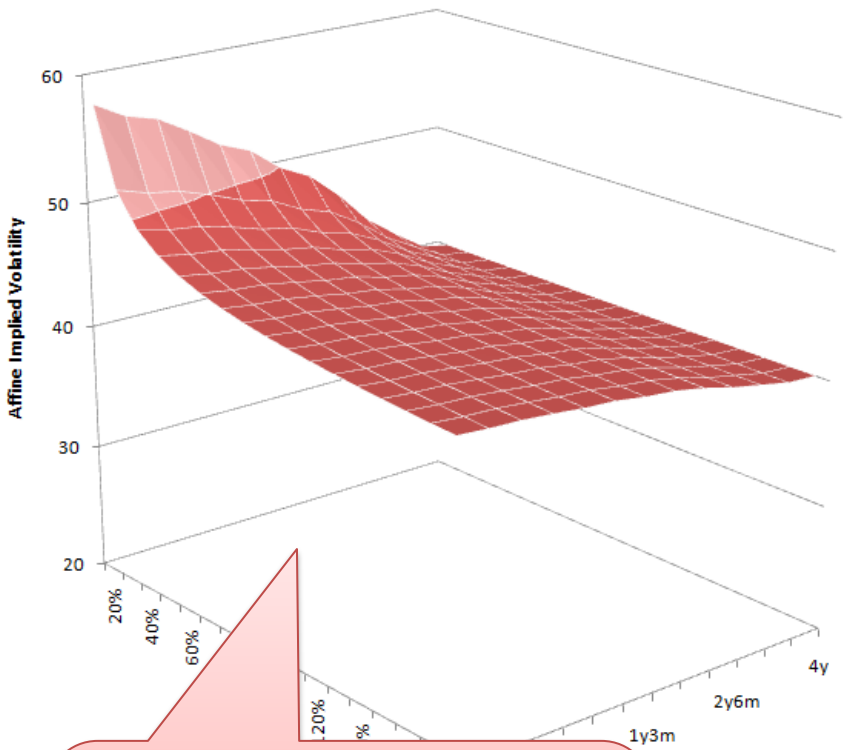
Implied Volatility Affine Dividends

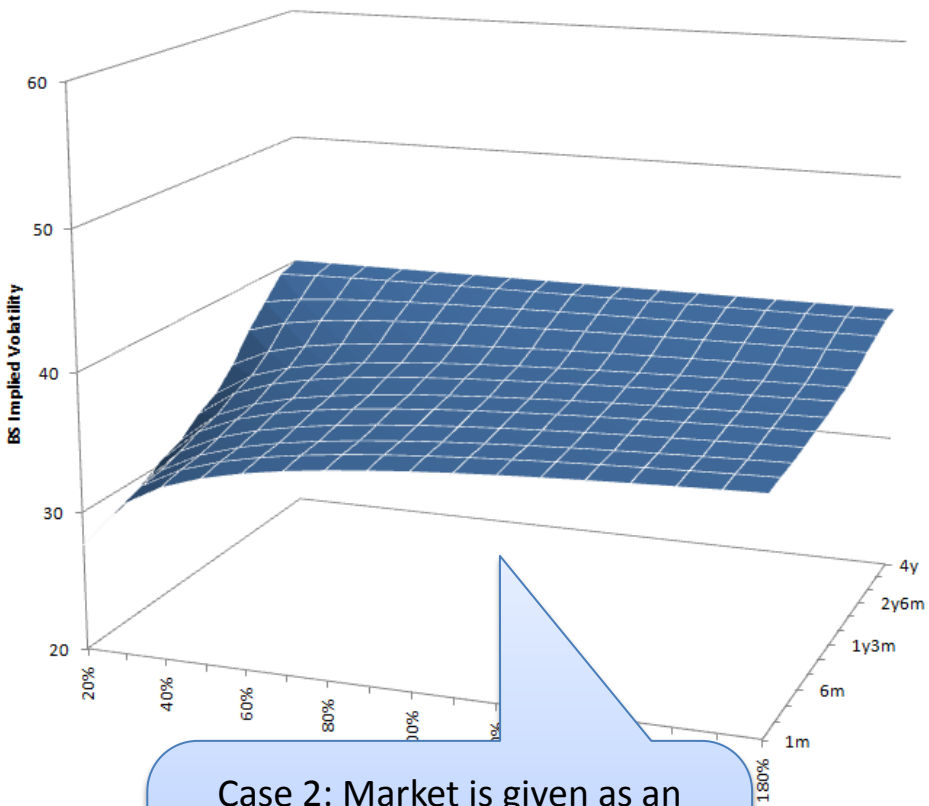
– The reverse interpretation allows us to convert observed market prices back into market prices on Z:

$$\mathrm{Call}_Z(T,\tilde{K}) := \frac{1}{\mathrm{DF}_T(F_T - A_T)}\,\mathrm{MarketCall}\left(T,(F_T - A_T)\tilde{K} + A_T\right)$$

which in turn allows us to compute Z's implied volatility from observed market data.

Implied Volatility and Dupire with Affine Dividends





Case 1: Market is given as a flat 40% BS world.
We imply the "pure" equity volatility for Z if we assume that dividends are cash for 3Y, then blended and purely proportional after 4Y

Case 2: Market is given as an affine dividend world with a 40% vol on Z (3Y cash, proportional after 4Y).
We imply the equivalent BS implied volatility for S.

Implied Volatility and Dupire with Affine Dividends

- Once we have the implied volatility from

$$\mathrm{Call}_Z(T,\tilde{K}) := \frac{1}{\mathrm{DF}_T(F_T - A_T)}\,\mathrm{MarketCall}\big(T,(F_T - A_T)\tilde{K} + A_T\big)$$

  we can compute Dupire's local volatility for stock prices with affine dividends as

$$\sigma_X(t,x)^2 := 2\frac{\partial_t \mathrm{Call}_X(t,x)}{x^2\partial_{xx}^2 \mathrm{Call}_Z(t,x)}$$

- Similarly, numerical methods are very efficient, see [2]
  - Simple credit risk
  - Variance Swaps with Affine Dividends
  - PDEs

Main practical issues

- Since the stock price depends on future dividends, any maturity-T option price has a sensitivity to any cash dividends past T.
- The assumption that a stock price keeps paying cash dividends even if it halves in value is not really realistic
  - Black & Scholes assumes at least that the dividend falls alongside the drop in spot price
  - Hence, assuming we are structurally long dividends it is more conservative on the downside to assume proportional dividends rather than cash dividends.

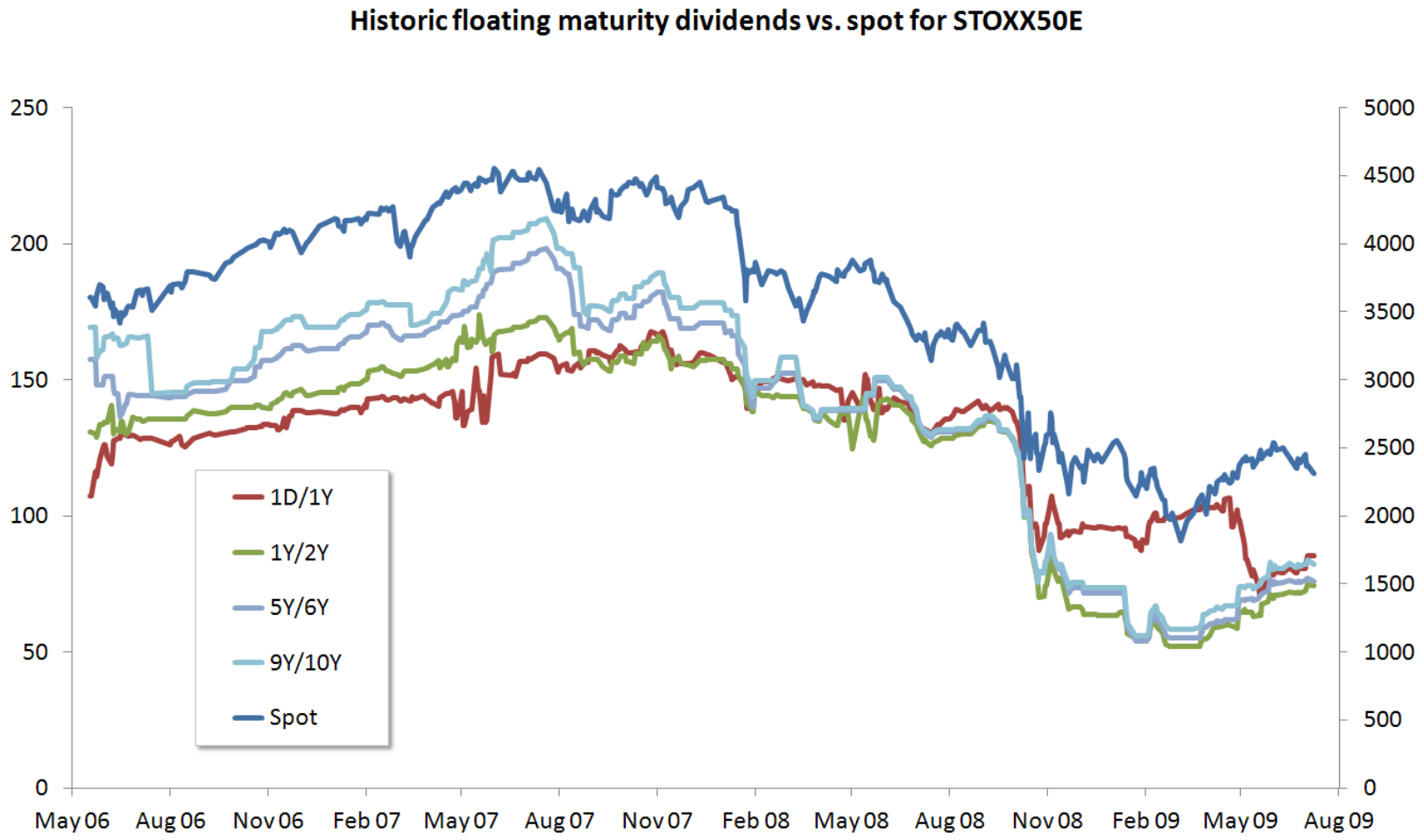
→ All in all, it would be desirable to have a dividend model which allows for spot dependency on the dividend level.

Part IV
Modeling Stochastic Dividends

Basics

- From the market of dividend swaps, we can imply a future level of dividends.
- The generally assumed behavior is roughly
  - The short end is “cash” (since rather certain)
  - The long end is “yield” (i.e. proportional dividends)

Dividends as an Asset Class



Modeling

- Before we looked at cash dividends.
  However, following our remarks before we can focus on the exponential formulation

$$S_{\tau_i} = S_{\tau_i -} - \Delta^i = S_{\tau_i -}(1 - e^{-d_i})$$

  - This "proportional dividend" approach makes life much easier – basically, to have a decent model, we "only" have to ensure that d remains positive.

- We will present a general framework for handling dividend models on 2F models.
- We start with a BS-type reference model

Modeling

- What do we want to achieve:
  - Very efficient model for test-pricing options on dividends
  - "Black-Scholes"-type reference model.
- Modeling assumptions
  - Deterministic rates (for ease of exposure)
  - We know the expected discounted *implied* dividends D and therefore the forward

$$F_t := R_t \left\{ S_0 - \sum_{k:\tau_k \le t} D_t^k \right\}$$

  - Our model should match the forward and
  - Drops at dividend dates

Proportional Dividends

– Black & Scholes with proportional discrete dividends:

$$d\log S_t = (r_t - \mu_t)dt + \sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt - \sum_i d_i \delta_{\tau_\kappa}(dt)$$

such that

$$S_t = \hat{F}_t \exp\left\{\int_0^t \sigma_s dW_s - \tfrac{1}{2}\int_0^t \sigma_s^2 ds\right\}$$

$$\hat{F}_t = \exp\left\{\int_0^t (r_s - \mu_s)dW_s - \sum_{k:\tau_k \le t} \hat{d}_k\right\}$$

to match the market forward we choose

$$d_k = -\log\left(1 - \frac{D_0^k}{F_{\tau_k -}}\right)$$

Proportional Dividends

– Since we always want to match the forward, consider the process

$$X_t := \log \frac{S_t}{F_t}$$

which has to have unit expectation in order to match the market.

- This approach has the advantage that we can take the explicit form of the forward out of the equation.
- In Black & Scholes, the result is

$$dX_t = \sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt$$

Stochastic Proportional Dividends (Buehler, Dhouibi, Sluys 2010 [3])

– Let u solve

$$du_t = -ku_t dt + \nu dB_t$$

and define

$$dX_t = \sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt - \sum_k (e_k u_{\tau_\kappa} + c_k)\delta_{\tau_\kappa}(dt)$$

- The volatility-like factor e_k expresses our (static) view on the dividend volatility:
  - $e_k = 1$ is the "normal"
  - $e_k = d_k$ is the "log-normal" case
- The constant c is used to calibrate the model to the forward, i.e. $\mathrm{E}[\exp(X_t)] = 1$.
- The deterministic volatility σ is used to match a term structure of option prices on S.

Modeling Stochastic Dividends

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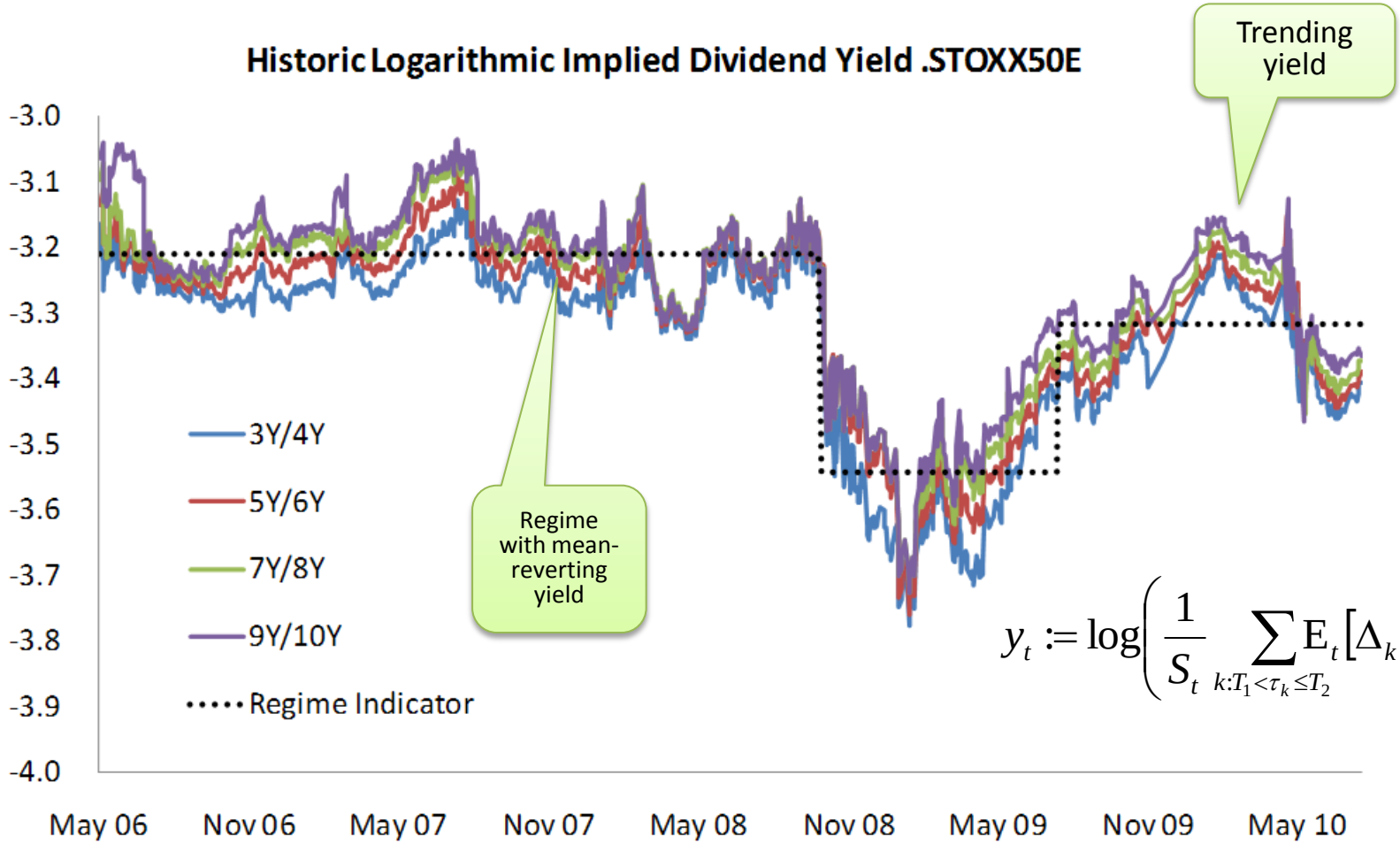


$$y_t := \log\left(\frac{1}{S_t}\sum_{k:T_1<\tau_k\le T_2} \mathrm{E}_t[\Delta_k]\right)$$

Stochastic Proportional Dividends

– We have

$$S_t = F_t \exp\left\{ \int_0^t \sigma_s dW_s - \tfrac{1}{2}\int_0^t \sigma_s^2 ds - \sum_{k:\tau_k \le t} \left(e_k u_{\tau_k} + c_k \right) \right\}$$

Note

- *log S/F* is normal → mean and variance of S are analytic.

– Step 1: Find c_k such that $\mathrm{E}[S_t] = F_t$.

– Step 2: Given the "stochastic dividend parameters" for u, find σ such that S reprices a term-structure of market observable option prices on S → model is perfectly fitted to a given strike range.

Stochastic Proportional Dividends

– Dynamics
The short-term dividend yield

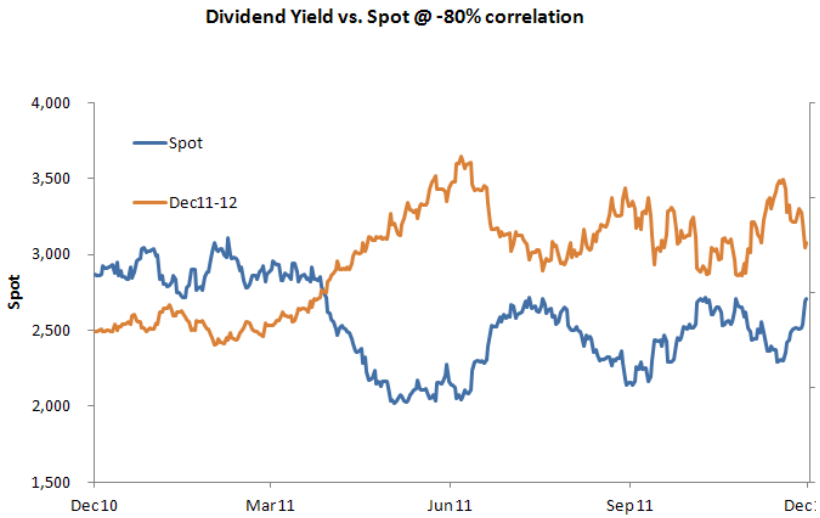
$$y_t := \log \frac{1}{S_t} \mathrm{E}_t \left[\Delta_{\tau_k} \right]$$

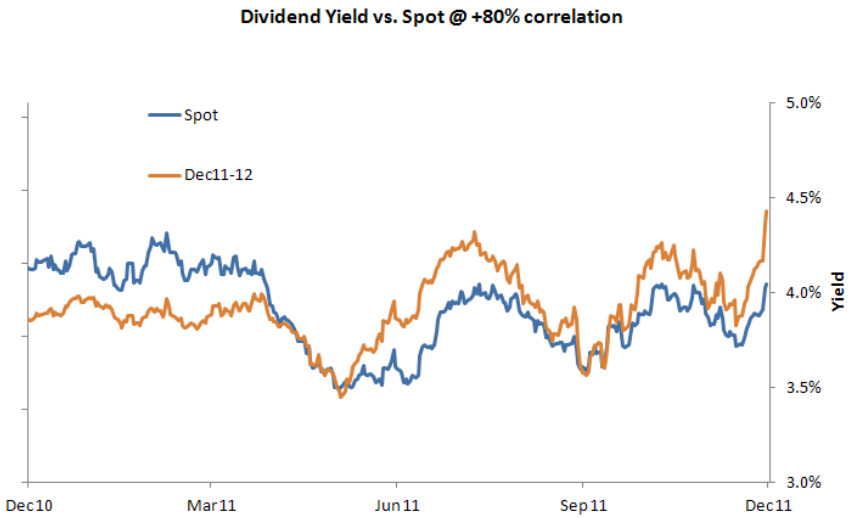
is approximately an affine function of u, i.e.

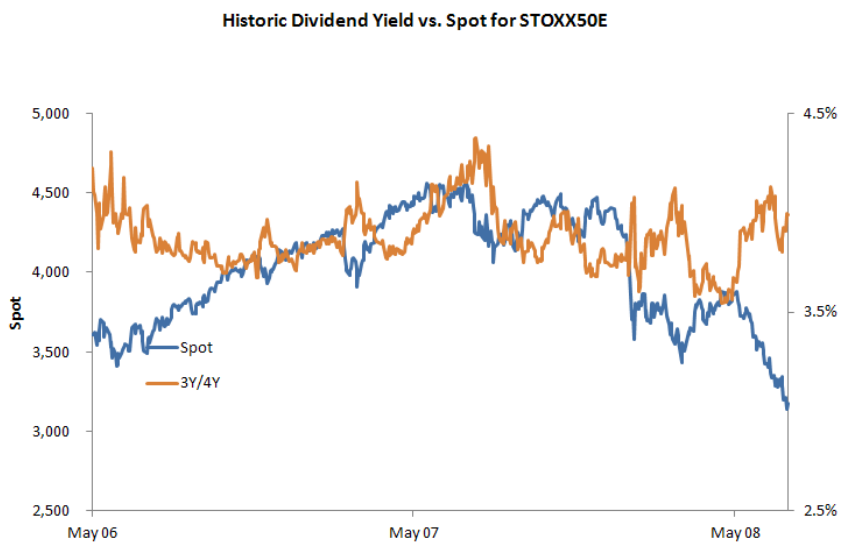
$$y_t \approx a + bu_t$$

- A strongly negative correlation therefore produces very realistic short-term behavior (nearly ‘fixed cash’) while maintaining randomness for the longer maturities.

Stochastic Proportional Dividends







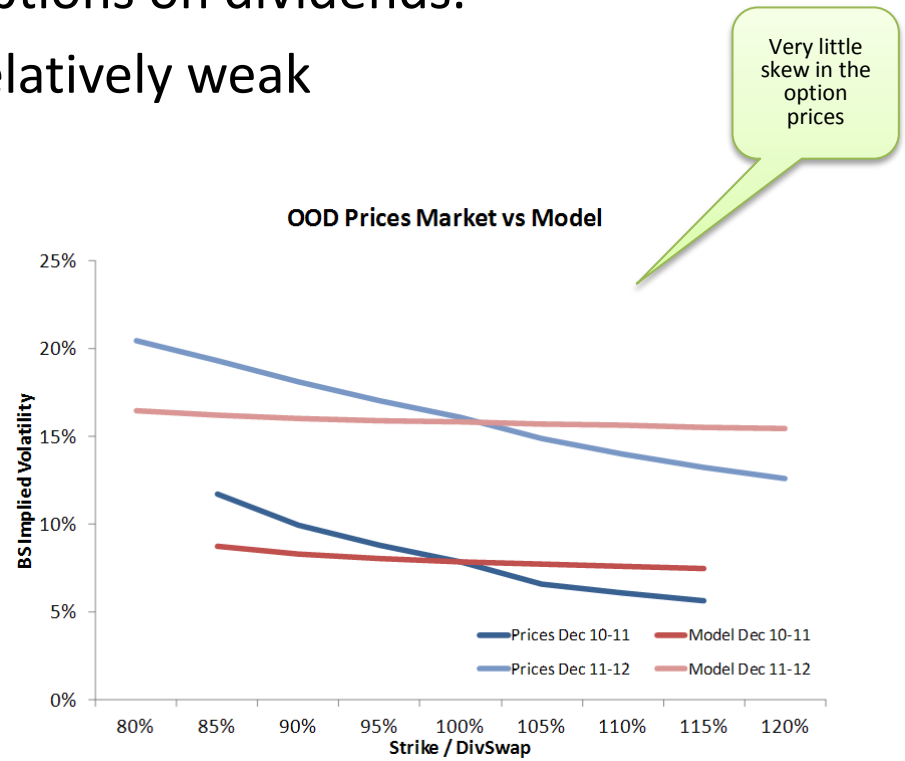
Stochastic Proportional Dividends

- Good
  - Very fast European option pricing → calibrates to vanillas
  - We can easily compute future forwards $\mathrm{E}_t[S_T]$ and therefore also future implied dividends.
  - Very efficient Monte-Carlo scheme with large steps since (X,u) are jointly normal.

Stochastic Proportional Dividends

- Not so great
  - Dividends *do* become negative
  - No skew for equity or options on dividends.
  - Dependency on stock relatively weak

→ try a more advanced version



From Stochastic Proportional Dividends to a General Model

$$dX_t = \sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt - \sum_k \left(d_k \phi(u_{\tau_k}, X_{\tau_k}) + c_k\right)\delta_{\tau_\kappa}(dt)$$

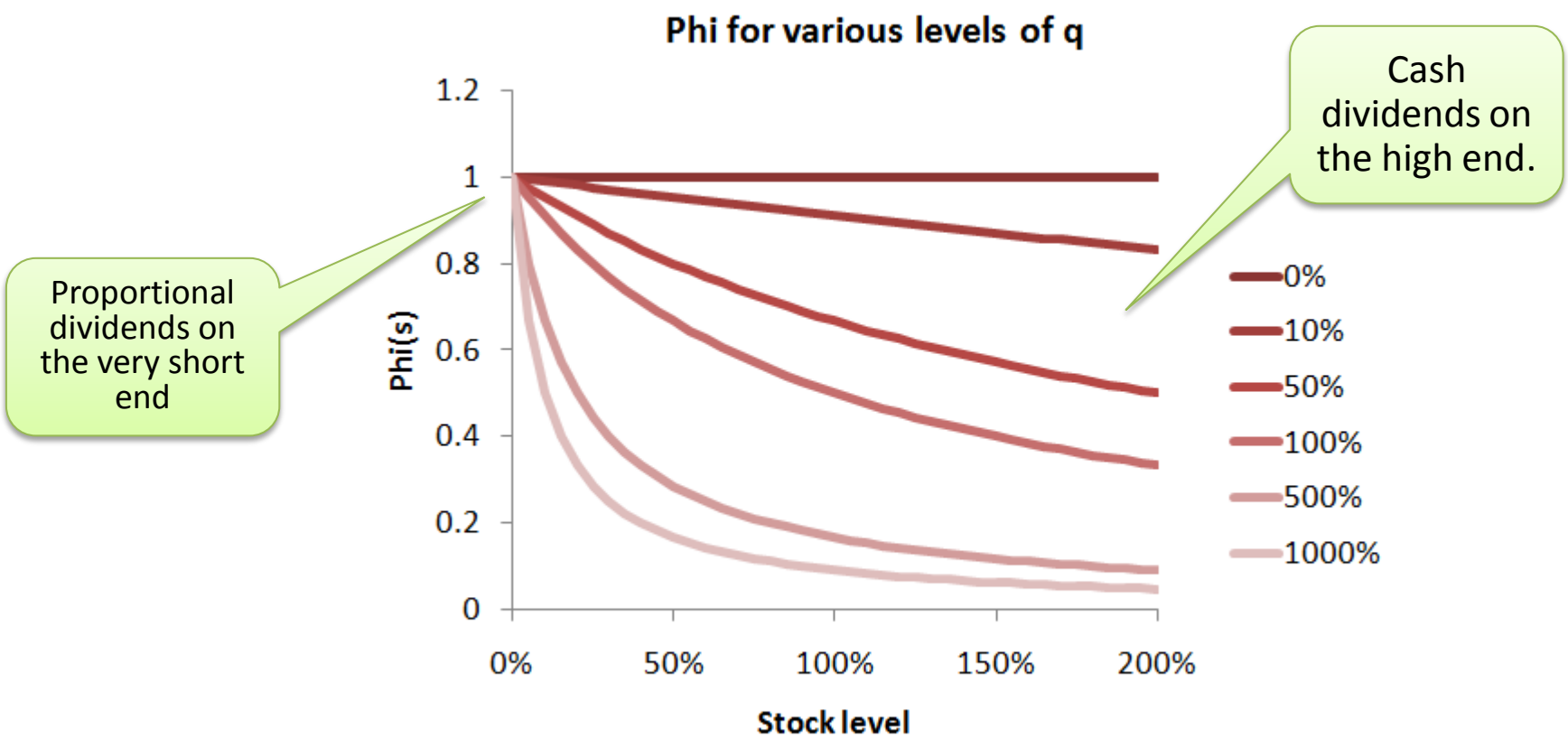
where this time we specify a convenient proportional factor function. A simple 1F choice

$$\phi(u,x) = \frac{1}{1+\theta e^x}$$

- $\theta = 1/S_*$ controls the dividend factor as a function spot:
  - For $S >> S_*$ we get $\phi \approx 1/S$ and therefore cash dividends.
  - For $S << S_*$ we get $\phi \approx 1$ and therefore yield dividends.
  - For $S = S_*$ we have a factor of ½. (nb the calibration will make sure that $\mathrm{E}[S] = F$).

From Stochastic Proportional Dividends to a General Model

$$\phi(u,x) = \frac{1}{1+\theta e^{x}}$$



From Stochastic Proportional Dividends to a General Model

– 2F version which avoids negative dividends

$$\phi(u,x) = \frac{1}{2}\frac{\alpha(\tanh(u)+1)}{1+\theta e^{x}}$$

various choices are available, but there are limits ... for example

$$\phi(u,x) = e^{-x}$$

yields a cash dividend model with absorption in zero.
→ future research into the allowed structure for ϕ.

Definition: Generalized Stochastic Dividend Model

- The general formulation of our new Stochastic Dividend Model is

$$
\begin{aligned}
dX_t = \quad & \sigma_t(X_t)dW_t - \frac{1}{2}\sigma_t^2(X_{t-})dt \\
& -(\phi_{\text{yield}}(u_t, X_{t-}) - c_t)dt \\
& -\sum_k \left(e_k \phi_{\text{discrete}}(u_{\tau_k}, X_{\tau_k -}) + c_k\right)\delta_{\tau_\kappa}(dt)
\end{aligned}
$$

- Note that following our "Exponential Representation Theorem" this model is actually very general:
  it covers *all* strictly positive two-factor dividend models where future dividends are Markov with respect to stock and another diffusive state factor ... in particular those of the form:

$$
dS_t = S_t \sigma_t(S_t)dW_t - \sum_k \Delta^k(S_{\tau_\kappa -}, u_t)\delta_{\tau_\kappa}(dt)
$$

  as long as $S>0$.

Generalized Stochastic Dividend Model

$$
\begin{aligned}
dX_t = \quad & \sigma_t(X_t)dW_t - \frac{1}{2}\sigma_t^2(X_{t-})dt \\
& -(\phi_{\text{yield}}(u_t, X_{t-}) - c_t)dt \\
& -\sum_k \left(e_k \phi_{\text{discrete}}(u_{\tau_k}, X_{\tau_k -}) + c_k\right)\delta_{\tau_\kappa}(dt)
\end{aligned}
$$

- This model allows a wide range of model specification including "cash-like" behavior if $\phi(u,s) \approx 1/s$.
- Compared to the Stochastic Proportional Dividend model, this model has the potential drawbacks that
  - Calibration of the fitting factors c is numerical.
  - Calculation of a dividend swap (expected sum of future dividends) conditional on the current state (S,u) is usually not analytic.
  - Spot-dependent dividends introduce Vega into the forward !!

The rest of the talk will concentrate on a general calibration strategy for such models using Forward PDEs.

Part V
Calibration

Calibrating the Generalized Stochastic Dividend model

- We aim to fit the model to both the market forward and a market of implied volatilities.
  The main idea is to use forward-PDE's to solve for the density and thereby to determine
  i. The drift adjustments c and
  ii. The local volatility σ.
- We assume that the volatility market is described by a "Market Local Volatility" which is implemented using the classic proportional dividend assumptions of Black & Scholes (or affine dividends).
- Discussion topics:
  - Forward PDE and Jump Conditions
  - Various Issues
  - Calibrating c and σ using a Generalized Dupire Approach
  - A few results.

Forward PDE

– Recall our model specification

$$dX_t = \sigma_t(X_t)dW_t - \frac{1}{2}\sigma_t^2(X_{t-})dt \qquad du_t = -\kappa u_t dt + \nu dB$$
$$-(\phi_{\text{yield}}(u_t, X_{t-}) - c_t)dt$$
$$-\sum_k \left(e_k \phi_{\text{discrete}}(u_{\tau_k}, X_{\tau_k -}) + c_k\right)\delta_{\tau_\kappa}(dt)$$

On $t \neq \tau_k$ this yields the forward PDE

$$\partial_t p_t(x,u) = \partial_x\left\{\left(\frac{1}{2}\sigma_t^2(x) + \phi_{\text{yield}}(t;x,u) + c_t\right)\cdot p_t(x,u)\right\} + \kappa\partial_u\{u \cdot p_t(x,u)\}$$
$$+ \frac{1}{2}\partial_{xx}^2\{\sigma_t^2(x)\cdot p_t(x,u)\} + \frac{1}{2}v^2\partial_{uu}^2\{p_t(x,u)\} + \rho v\partial_{xu}^2\{\sigma(x)\cdot p_t(x,u)\}$$

with the following jump condition on each dividend date:

$$p_{\tau_\kappa}(x,u) = p_{\tau_\kappa -}(x + \phi_{\text{discrete}}(x,u), u)$$

Issue #1: Jump Conditions

- The jump conditions require us to interpolate the density p on the grid which is an expensive exercise – hence, we will approximate the dividends by a "local yield".
- This simply translates the problem into a convection-dominance issue which we can address by shortening the time step locally (in other words, we are using the PDE to do the interpolation for us).
  - Definitely the better approach for Index dividends.

Issue #2: Strong Cross-Terms

- The most common approach to solving 2F PDEs is the use of ADI schemes where we do a θ-step in first the x and then the u direction and alternate forth.
  - The respective other direction is handled with an explicit step – and that step also includes the cross derivative terms.
  - If $|\rho| \rightarrow 1$, this becomes very unstable and ADI starts to oscillate ... in our cases, a strongly negative correlation is a sensible choice.
- We therefore employ an Alternating Direction Explicit "ADE" scheme as proposed by Dufffie in [9].

ADE Scheme

- Assume we have a PDE in operator form

$$\frac{dp}{dt} = Ap$$

- we split the operator $A=L+U$ into a lower triangular matrix L and an upper triangular matrix U, where each carries half the main diagonal.
- Then we alternate implicit and explicit application of each of those operators:

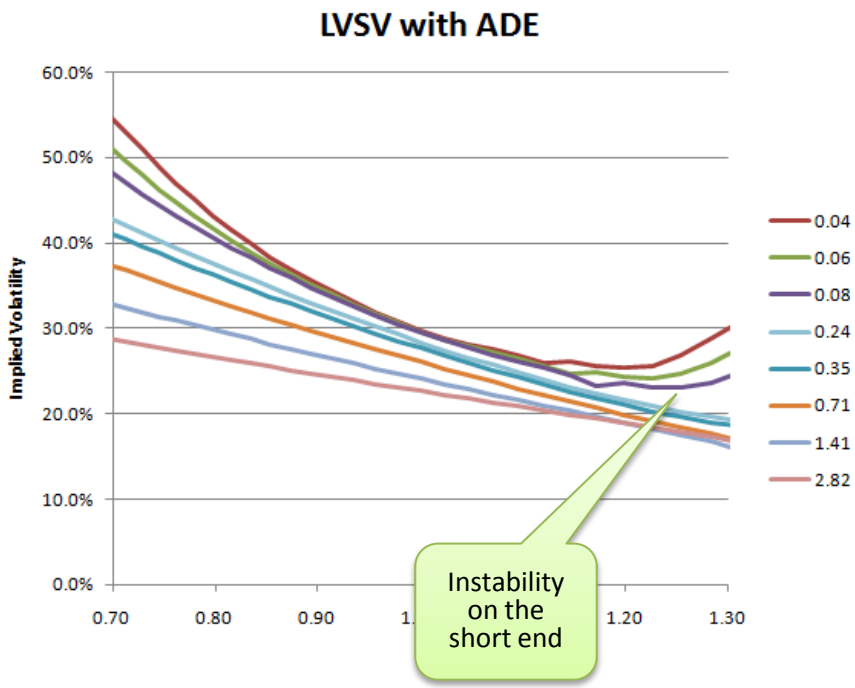
$$p_{t+dt/2} - p_t = dtLp_{t+dt/2} + dtUp_t$$

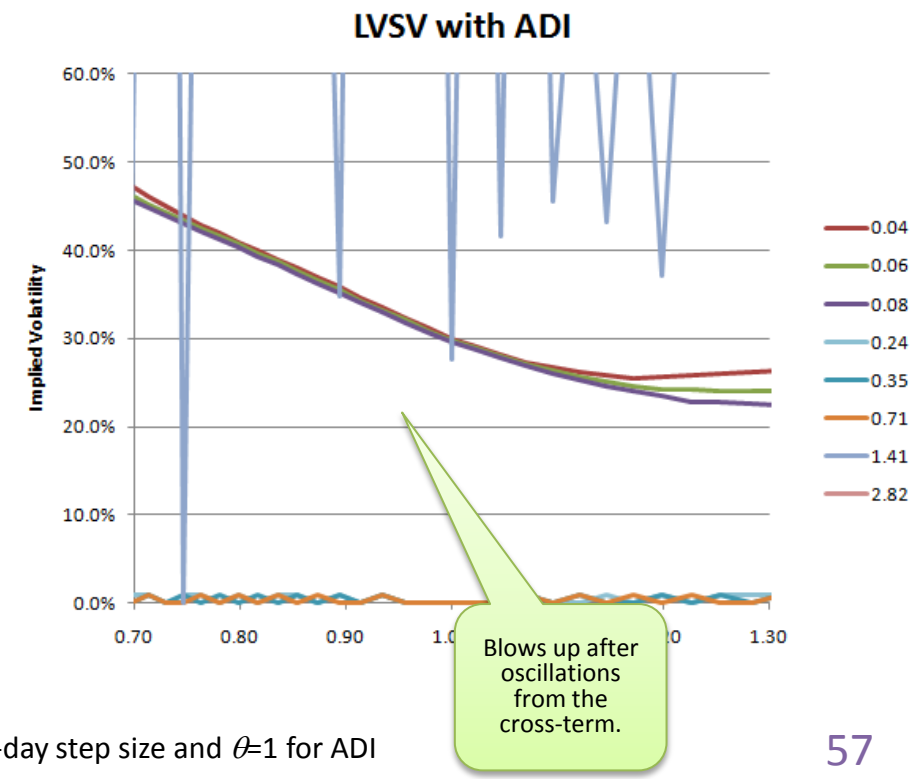
$$p_{t+dt} - p_{t+dt/2} = dtLp_{t+dt/2} + dtUp_{t+dt}$$

- However, since both U and L are tridiagonal, solving the above is actually *explicit* – hence the name.
  - This scheme is unconditionally stable and therefore good choice for problems like the one discussed here.
  - In our experience, the scheme is more robust towards strongly correlated variables ... and much faster for large mesh sizes.
  - However, ADI is better if the correlation term is not too severe.

ADE vs. ADI

- Stochastic Local Volatility $dS_t = \sigma(t; S_t) e^{\frac{1}{2}u_t} S_t dW_t$ where we additionally cap and floor the total volatility term.
- In the experiments below, the OU process parameters where κ=1, ν=200% and correlation ρ=-0.9 (*).



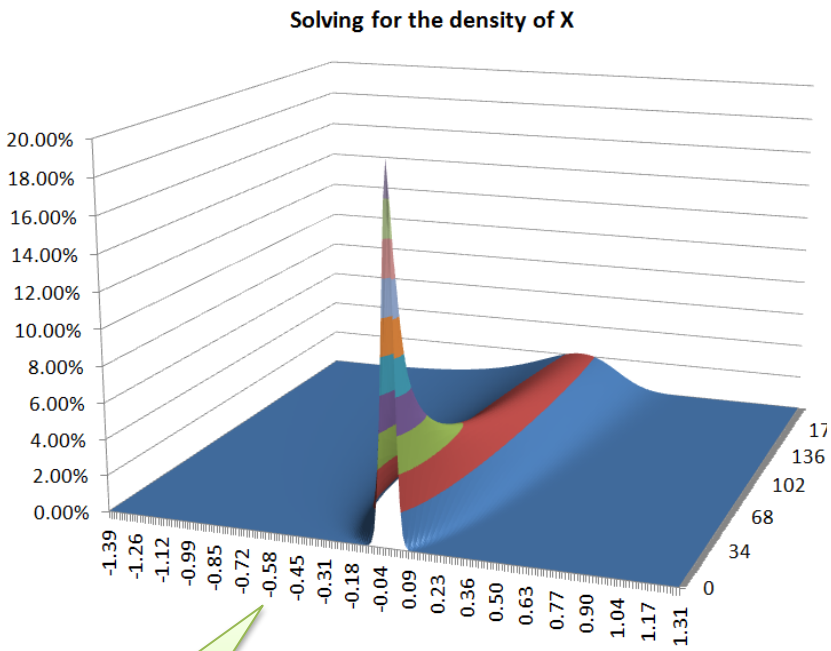


(*) we used X=log$S$, not scaled. Grid was 401x201 on 4 stddev with a 2-day step size and θ=1 for ADI

Issue #3: Grid scaling

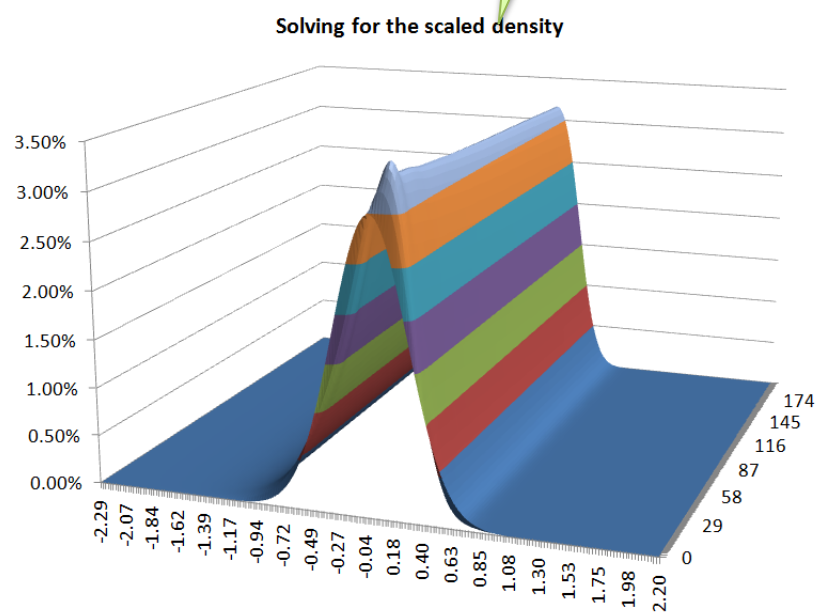
- We wish to calibrate our joint density for both short and long maturities from, say, 1M up to 10Y.
- A classic PDE approach would mean that we have to stretch our available mesh points sufficiently to cover the 10Y distribution of our process ... but then the density in the short end will cover only very few mesh points.
- The basic problem is that the process X in particular expands with $\mathrm{sqrt}(t)$ in time (u is mean-reverting and therefore naturally 'bound').
- We follow Jordinson in [1] and scale both the process X and the OU process u by their variance over time → this gives (in the no-skew case) a constant efficiency for the grid.

Jordinson Scaling



Imprecise for short maturities

Constant precision over the entire time line



Jordinson Scaling

– It is instructive to assess the effect of scaling X. Since X follows

$$dX_t = \sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt - \delta_t\left(\phi(u_t, X_t) + c_t\right)dt$$

we get

$$d\tilde{X}_t = \frac{\sigma_t}{\sqrt{t}} dW - \frac{1}{\sqrt{t}}\left\{\frac{1}{2}\sigma_t^2 dt + \delta_t\left(\phi(u_t, \tilde{X}_t\sqrt{t}) + c_t\right)\right\}dt - \frac{1}{t}\tilde{X}_t dt$$

→ the rather ad-hoc solution is to start the PDE in a state dt where this effect is mitigated.

Very strong convection dominance for t→0

Calibration

- Let us assume that our forward PDE scheme converges robustly.
- The next step is to use it to calibrate the model to the forward and volatility market.

Generalized Dupire Calibration

- Assume we are given
  - A state process u with known parameters.
  - A jump measure J with finite activity (e.g. Merton-type jumps; dividends; credit risk ...) and jumps $\omega_t(S_{t-},u_t)$ which are distributed conditionally independent on F_{t-} with distribution $q_t(S_{t-},u_t;\cdot)$

- We aim at the class of models of the type

The drift c will be used to fit the forward to the market

$$
\begin{aligned}
dS_t &= (\mu(t;S_{t-},u_t)-c_t)S_{t-}dt+\sigma(t;S_{t-})\zeta(t;u_t)S_{t-}dW_t \\
&\quad -S_{t-}(1-e^{\omega_t(S_{t-},u_t)})J_t(dt;S_{t-},u_t) \\
du_t &= \alpha(\cdots)dt+\beta(\cdots)dW_t^v+\cdots
\end{aligned}
$$

Note the separable volatility.

- where we wish to calibrate
  - c to fit the forward to the market i.e. $\mathrm{E}[S_t] = F_t$.
  - σ to fit the model to the vanilla option market – we assume that this is represented by an existing Market Local Volatility Σ.

Generalized Dupire Calibration

– Let us also introduce m such that

$$F_t = S_0 \exp\left(\int_0^t m_s ds\right)$$

where m may have Dirac-jumps at dividend dates.

– We will also look at the un-discounted option prices

$$C_{\text{market}}(t,K) := \frac{1}{DF(t)} Call(t,K)$$

and for the model

$$C_{\text{model}}(t,K) := \mathrm{E}\left[(S_t - K)^+\right]$$

Generalized Dupire Calibration - Examples

Stochastic interest rates, see Jordinson in [1]

$$dS_t = (r(t;u_t) - c_t)S_t dt + \sigma(t;S_t)S_t dW$$

Stochastic local volatility c.f. Ren et al [7]

$$dS_t = m_t S_t dt + \sigma(t;S_t)e^{\frac{1}{2}u_t} S_t dW_t$$

Merton-type jumps

$$dS_t = m_t S_t dt + \sigma(t;S_{t-})S_{t-}dW_t + \lambda_t \mathrm{E}[e^{-\xi_t} - 1]\mathrm{dt} - \sum_k S_{t-}(1 - e^{-\xi_t})N_\lambda(dt)$$

$$dS_t = m_t S_t dt + \sigma(t;S_t)S_{t-}dW_t + \lambda_t S_{t-}^{-p} dt - S_{t-}dN_{\lambda_t S_t^{-p}}(dt)$$

Default risk modeling with state-dependent intensity a'la Andersen et al [8]

We used N_λ to indicate a Poisson-process with intensity λ.

Calibration

$$
\begin{aligned}
dS_t &= (\mu(t;S_{t-},u_t)-c_t)S_{t-}dt+\sigma(t;S_{t-})\zeta(t;u_t)S_{t-}dW_t \\
&\quad -S_{t-}(1-e^{\omega_t(S_{t-},u_t)})J_t(dt;S_{t-},u_t) \\
du_t &= \alpha(\cdots)dt+\beta(\cdots)dW_t^v+\cdots
\end{aligned}
$$

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Generalized Dupire Calibration

- We apply Ito to a call price and take expectations to get

$$
\begin{aligned}
\frac{1}{dt}\mathrm{E}\left[d(S_t-K)^+\right] &= \mathrm{E}\left[S_{t-}1_{S_{t-}>K}\mu(t;S_{t-},u_t)\right]-c(t)\mathrm{E}\left[S_{t-}1_{S_{t-}>K}\right] \\
&\quad +\frac{1}{2}K^2\sigma(t;K)^2\mathrm{E}\left[\delta_{S_{t-}=K}\zeta(t;u_t)^2\right] \\
&\quad +\mathrm{E}\left[\left\{\left(S_{t-}e^{-\omega_t(S_{t-},u_t)}-K\right)^+-\left(S_{t-}-K\right)^+\right\}J_t(dt;S_{t-},u_t)\right]
\end{aligned}
$$

- If the density of (S,u) is known at time t-, then all terms on the right hand side are known except σ and c.
  - If c is fixed, then we have independent equations for each K.

- The left hand side is the change in call prices in the model.
  - The unknown there is

$$\mathrm{E}\left(S_{t+dt}-K\right)^+$$

Generalized Dupire Calibration – Drift

- In order to fix c, we start with the case $K=0$: we know that the zero strike call in the market satisfies

$$\frac{1}{dt} C_{market}(t,0) = m_t F_t dt$$

On the other hand, our equation shows that

$$\frac{1}{dt} d\mathrm{E}[S_t] = \mathrm{E}\left[S_{t-} 1_{S_{t-}>K} \mu(t;\cdot)\right] - c(t)\mathrm{E}\left[S_{t-} 1_{S_{t-}>K}\right] + \mathrm{E}\left[S_{t-}(e^{-\omega_t(\cdot)} - 1) J_t(dt;\cdot)\right]$$

- hence we have two options to determine the left hand side:
  a. Incremental Fit:

$$d\mathrm{E}[S_t] \overset{!}{=} m_t F_t dt$$

  b. Total Fit :

$$\mathrm{E}[S_{t+dt}] \overset{!}{=} F_{t+dt}$$

Generalized Dupire Calibration – Drift

– Using the "Total Fit" approach is much more natural since it uses c to make sure that

$$dF_{t+dt} = \mathrm{E}[S_t] + \mathrm{E}\left[S_{t-}1_{S_{t-}>K}\,\mu(t;\cdot)\right]dt - c(t)\mathrm{E}\left[S_{t-}1_{S_{t-}>K}\right]dt$$

which is a primary objective of the calibration.

- The "Incremental Match" suffers from numerical instability: if the fitting process encounters a problem and ends up in a situation where $\mathrm{E}[S_t] \neq F_t$, then fitting the differential $\mathrm{dE}[S_t]$ will not help to correct the error.
- The "Total Match", on the other hand, will start self-correcting any mistake by "pulling back" the solution towards the correct $\mathrm{E}[S_{t+dt}] = F_{t+dt}$. However, depending on the severity of the previous error, this may lead to a very strong drift which may interfere with the numerical scheme at hand.
- The optimal choice is therefore a weighting between the two schemes.

Generalized Dupire Calibration – Volatility

Recall

$$\begin{aligned}\frac{1}{dt}C_{\text{model}}(t,K) &= \mathrm{E}\left[S_{t-}1_{S_{t-}>K}\,\mu(t;S_{t-},u_t)\right]-c(t)\mathrm{E}\left[S_{t-}1_{S_{t-}>K}\right]\\ &\quad+\frac{1}{2}K^2\sigma(t;K)^2\mathrm{E}\left[\delta_{S_{t-}=K}\zeta(t;u_t)^2\right]\\ &\quad+\mathrm{E}\left[\left\{\left(S_{t-}e^{-\omega_t(S_{t-},u_t)}-K\right)^+-\left(S_{t-}-K\right)^+\right\}J_t(dt;S_{t-},u_t)\right]\end{aligned}$$

– where we now have determined the drift correction c - this leaves us with determining the local volatility $\sigma(t,K)$ for each strike K.

We have again the two basic choices regarding $dC(t,K)$:

a. Incremental match (essentially Ren/Madan/Quing [7] 2007 for stochastic local volatility):

$$\frac{1}{dt}C_{\text{model}}(t,K)\overset{!}{=}\frac{1}{dt}C_{\text{market}}(t,K)$$

b. Total Match (Jordinson mentions for his rates model in [1] 2006):

$$C_{\text{model}}(t+dt,K)\overset{!}{=}C_{\text{market}}(t+dt,K)$$

Generalized Dupire Calibration – Volatility

– The incremental match

$$\frac{1}{dt} C_{\text{model}}(t,K) \overset{!}{=} \frac{1}{dt} C_{\text{market}}(t,K)$$

– It has the a nice interpretation in the case where we calibrate a stochastic local volatility model.

- The market itself satisfies

$$\frac{dC_{market}(t,K)}{dt} = \frac{1}{2} K^2 \sigma_{market}(t,K)^2 \mathrm{E}\left[\delta_{S_t=K}\right]$$

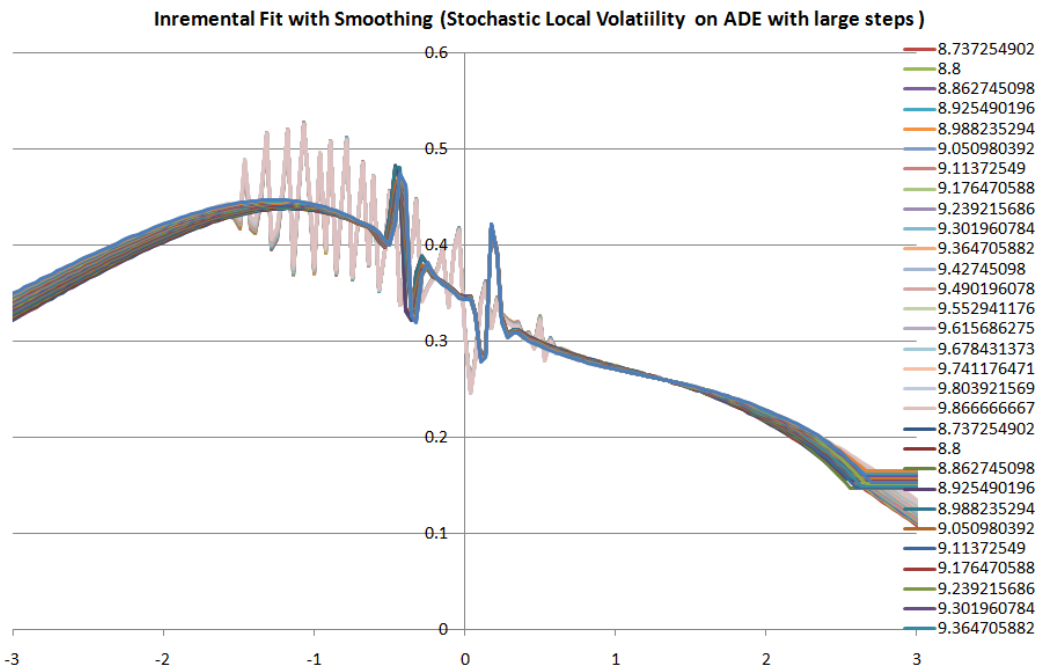
hence we can set

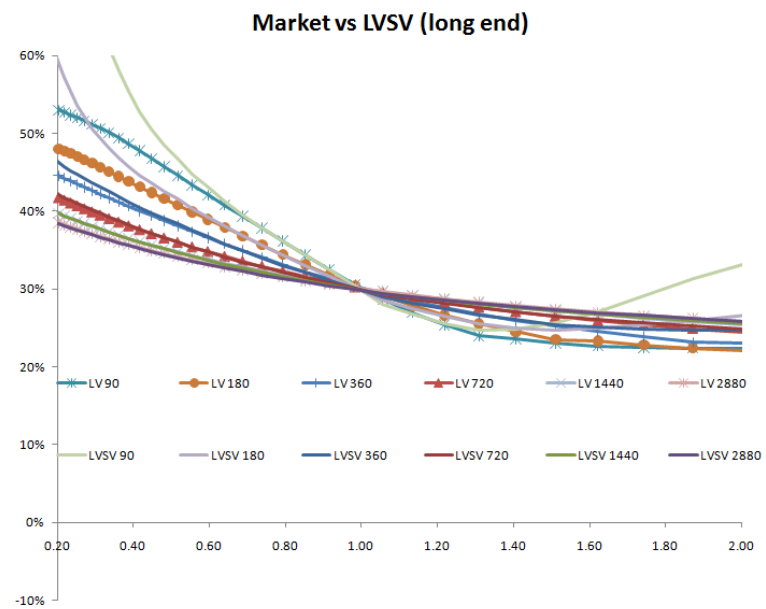
$$\sigma(t;K)^2 := \sigma_{\text{market}}(t;K)^2 \frac{\mathrm{E}\left[\delta_{S_{t-}=K}\right]}{\mathrm{E}\left[\delta_{S_{t-}=K}\zeta(t;u_t)^2\right]}$$

Generalized Dupire Calibration – Volatility

– Good & bad for the Incremental Fit:

- This formulation suffers from the same numerical drawback of calibrating to a "difference" as we have seen for c: it does not have the power to pull itself back once it missed the objective.
  It suffers from the presence of dividends (if the original market is given by a classic Dupire LV model) or numerical noise.
- The upside of this approach is that it produces usually smooth local volatility estimates for stochastic local volatility and yield dividend models.





Generalized Dupire Calibration – Volatility

- The total match following a comment of Jordinson in [7] 2007 for stochastic rates means to essentially use

$$C_{\text{model}}(t+dt,K) \overset{!}{=} C_{\text{market}}(t+dt,K)$$

- Good & Bad
  - As in the c-calibration case, it has the desirable "self-correction" feature which makes it very suitable for models with dividends which suffer usually from the problem that the target volatility surface is not produces consistently with the respective dividend assumptions.
    - It also helps to iron out imprecision arising from the use of an imprecise PDE scheme.
  - The downside is that the self-correcting feature is a local operation.
    It can therefore lead to highly non-smooth volatilities which in turn cause issues for the PDE engine.
    - We therefore chose to smooth the local volatilities after the total fitting with a smoothing spline.

Calibration

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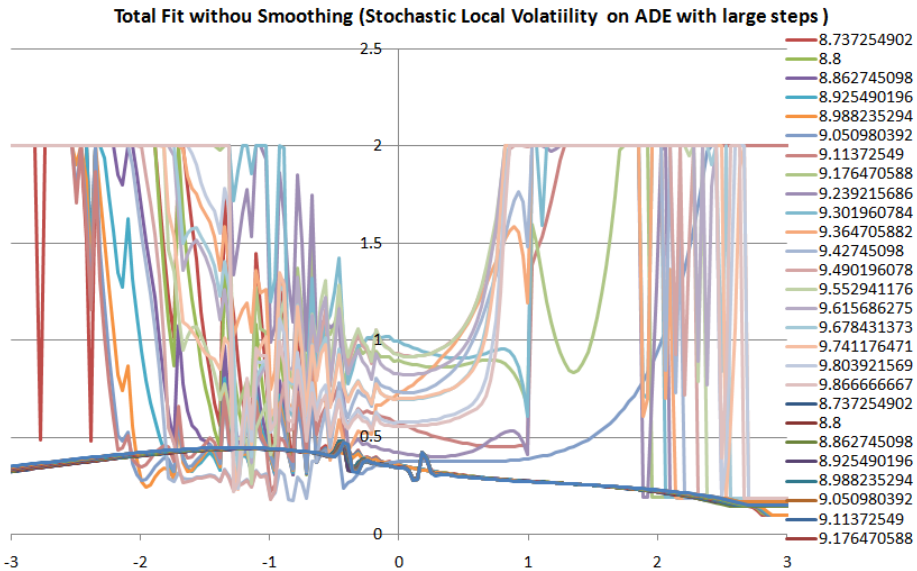

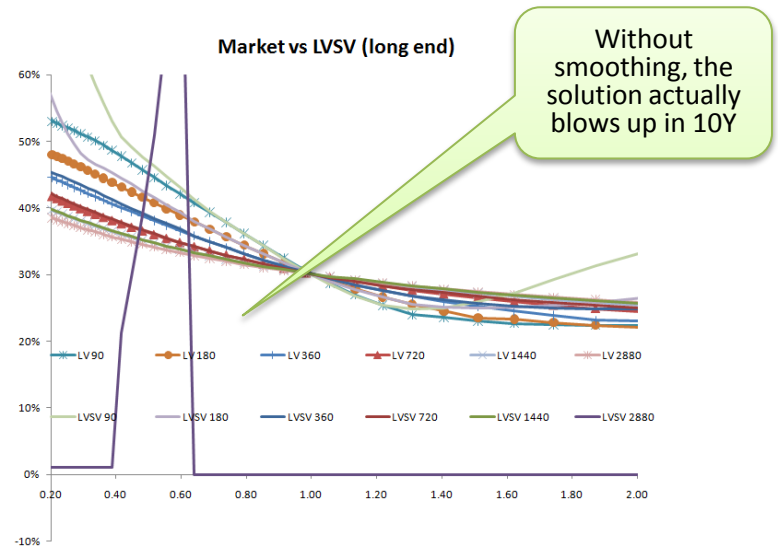

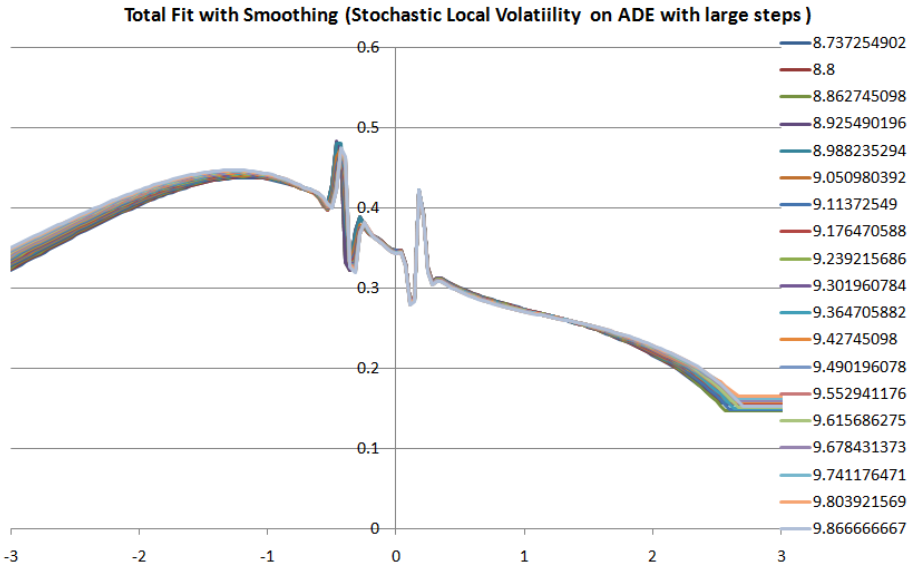

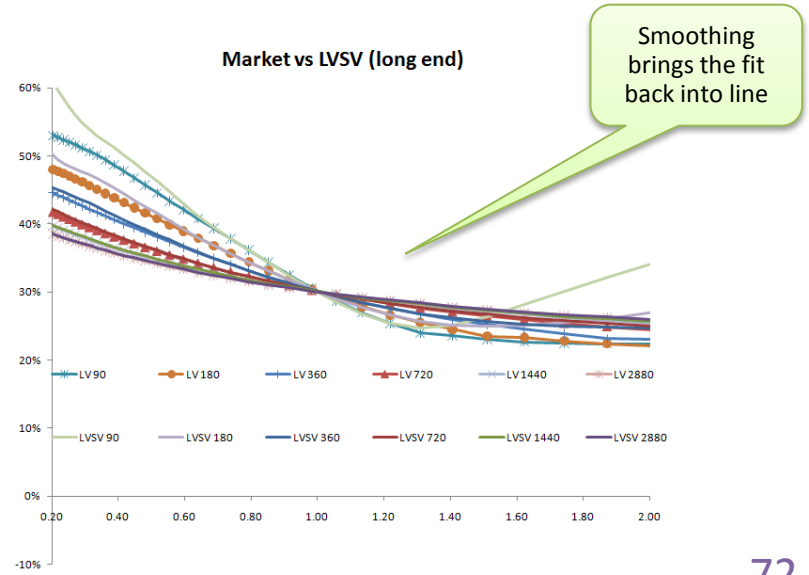

Calibration

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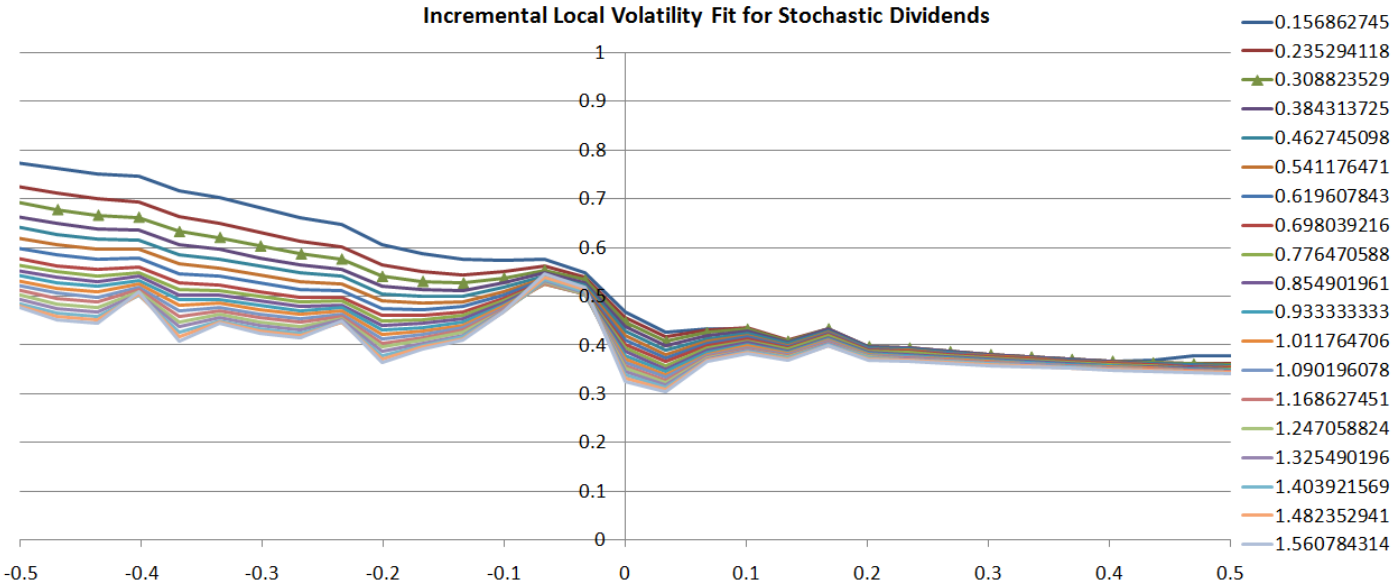

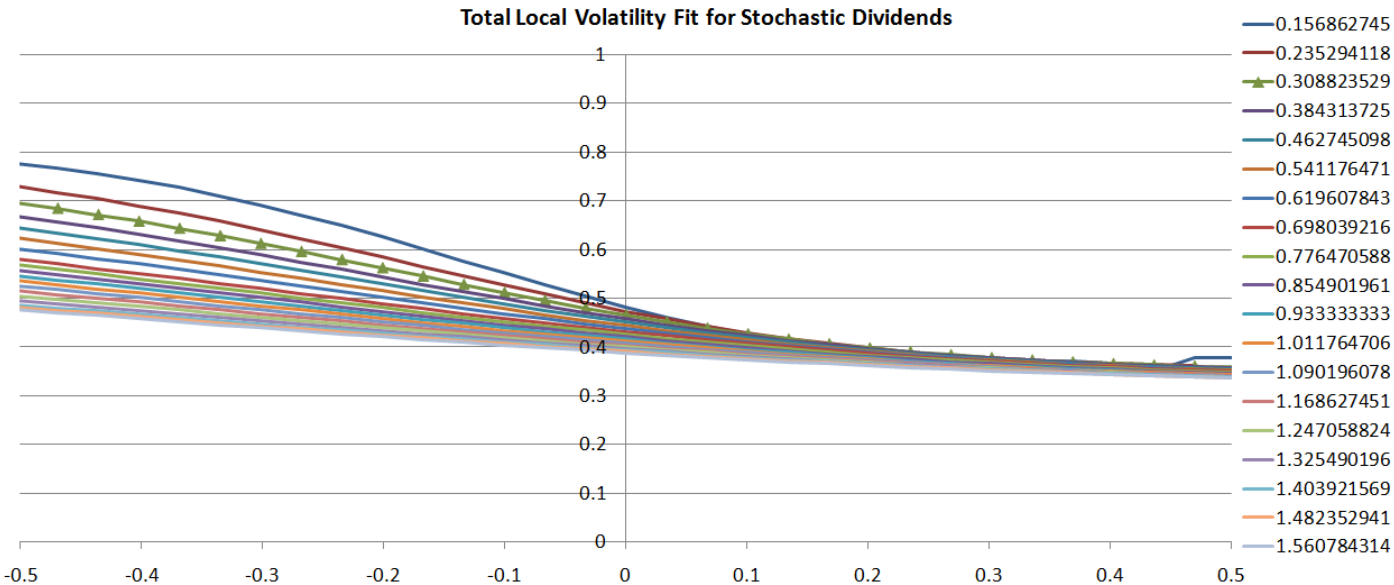

Generalized Dupire Calibration - Summary

– Any model of the type

$$
\begin{aligned}
dS_t &= (\mu(t;S_{t-},u_t)-c(t))S_{t-}dt+\sigma(t;S_{t-})\zeta(t;u_t)S_{t-}dW_t \\
&\quad -S_{t-}(1-e^{\omega_t(S_t,u_t)})J_t(dt;S_{t-},u_t) \\
du_t &= \alpha(\cdots)dt+\beta(\cdots)dW_t^v+\cdots
\end{aligned}
$$

can very efficiently be calibrated using forward-PDEs.

- First fit c to match the forward with incremental fitting
- Match σ with a mixture of incremental and total fitting.
- Apply smoothing to the local volatility surface to aid the numerical solution of the forward PDE.
- The calibration time on a 2F PDE with ADE/ADI is negligible compared to the evolution of the density → we can do daily calibration steps.
- Index dividends are transformed into yield dividends.

Last Slide

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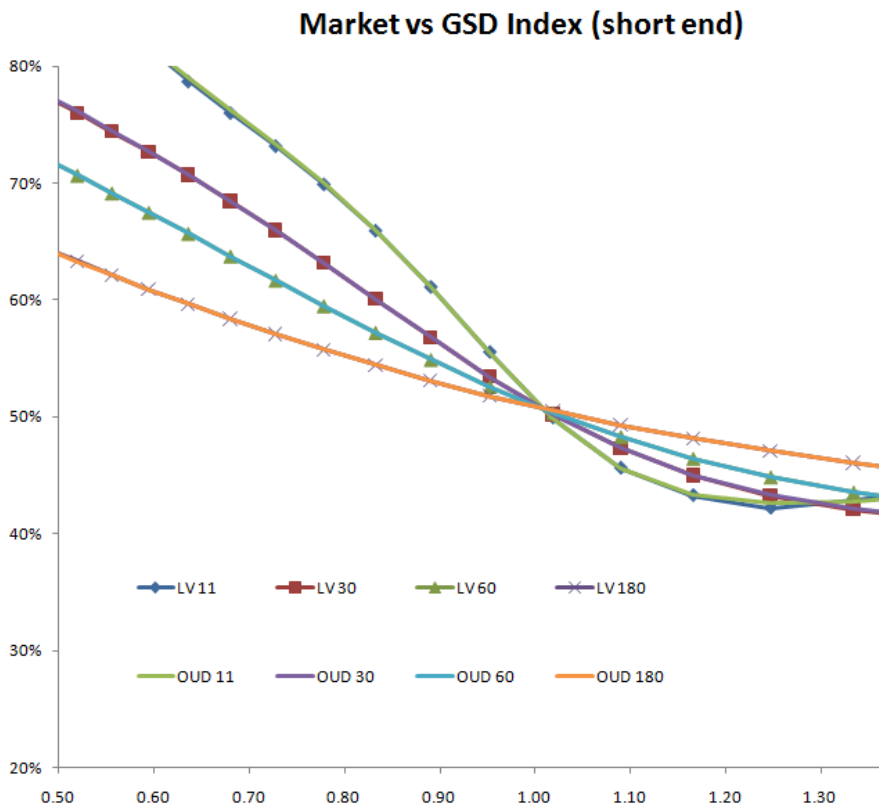
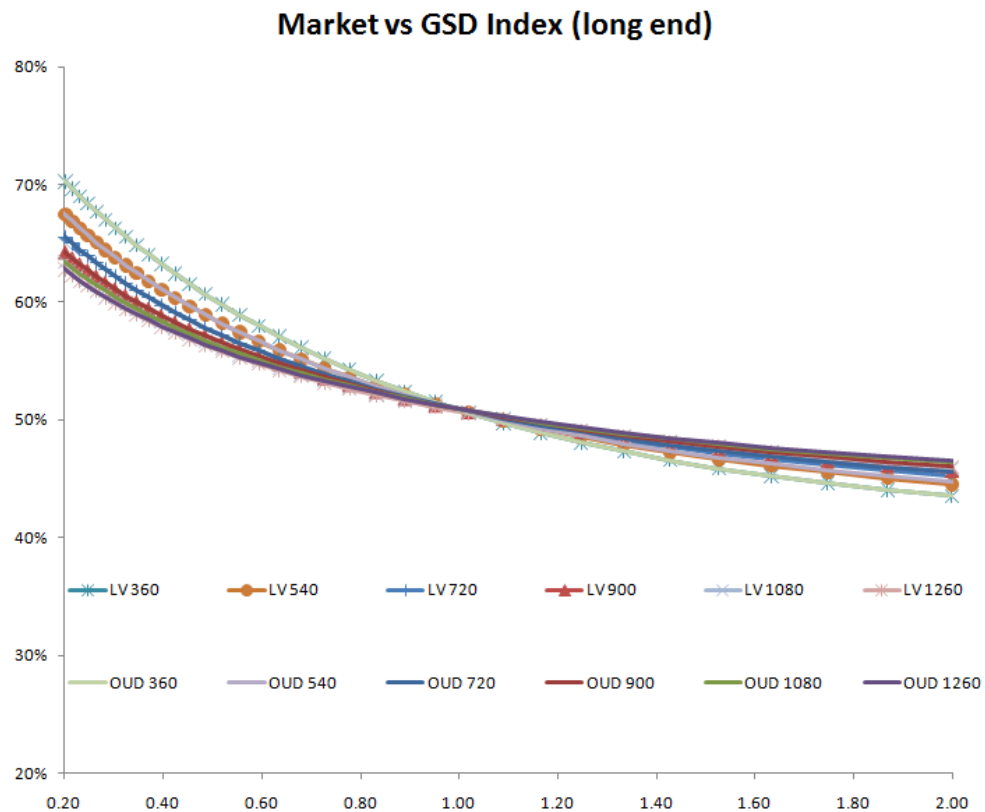
Generalized Stochastic Dividend Model (Index version)





$$dX_t = \sigma_t(X_t)dW_t - \frac{1}{2}\sigma_t^2(X_{t-})dt - (\phi_{\text{yield}}(u_t, X_{t-}) - c_t)dt$$

Thank you very much for your attention

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