Options On Variance: Pricing And Hedging

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Options On Variance: Pricing And Hedging

Introduction

- This talk is on pricing options on *realised variance*, which is defined over business days $T_1 = t_0 < \cdots < t_n = T_2$ as

$$RV(T_1, T_2) := \sum_{i=1}^{n} \left(\log \frac{S_{t_t}}{S_{t_{t-1}}} \right)^2$$

  - The scaled quantity $1/n\ RV(T_1, T_2)$ is an unbiased estimator for the variance in $[T_1, T_2]$ if we neglect any drift.
  - Second-order taylor on the log contract gives

Static position in Log-contracts

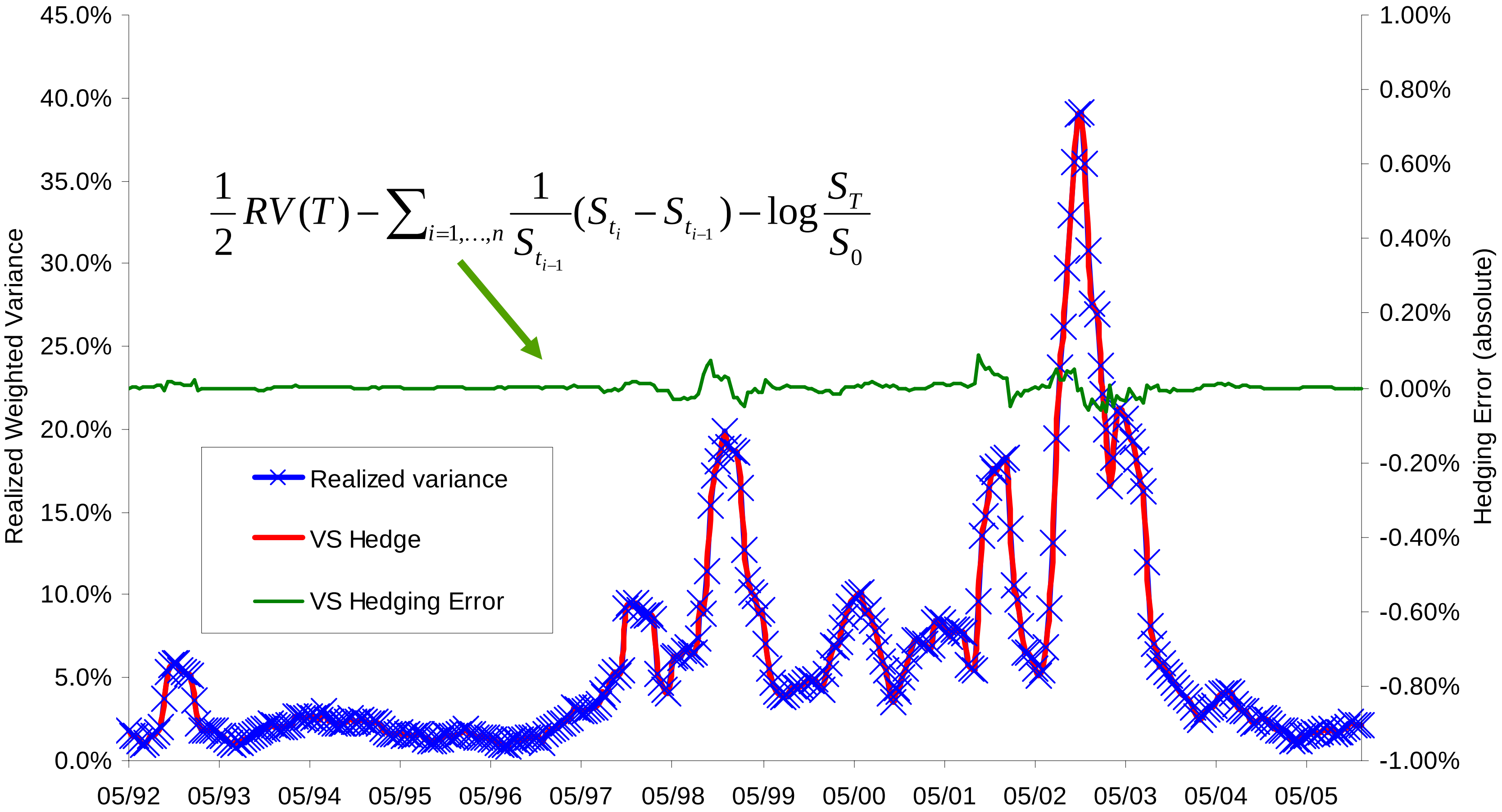
$$\frac{1}{2} RV(T_1, T_2) \approx \sum_{i=1,\ldots,n} \frac{1}{S_{t_{i-1}}} (S_{t_i} - S_{t_{i-1}}) - \log \frac{S_{T_2}}{S_{T_1}}$$

Delta-Hedging

Options On Variance: Pricing And Hedging

Introduction

.STOXX50E Realized Variance Hedge (90 days)



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Options On Variance: Pricing And Hedging

Introduction

- The purpose of this talk is to line out ways to evaluate non-linear payoffs on realized variance
  - Vanilla options on variance (*)

$$\left(RV(T_1,T_2)-K^2\right)^+ ,\ \left(\sqrt{RV(T_1,T_2)}-K\right)^+$$

  - Forward started options and options on volatility indices (VIX, VDAX$_{new}$)

$$\left(RV(T_1,T_2)-kV_{T_1}(T_1,T_2)\right)^+$$

  - „Hybrids“

$$\left(RV(T_1,T_2)-V_{T_1}(T_1,T_2)\right)^+ 1_{S_{T_1}<B}$$

$V_{T1}(T_1,T_2)$: Market value of $RV(T_1,T_2)$ at time $T1$ (≈ variance swap)

(*) All payoffs in this presentation are in “natural” scale. Business day adjustments might be applied in practise.

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Options On Variance: Pricing And Hedging

Questions

- Realized variance:
  Dividends, defaults and its relation to quadratic variation.
- Pricing:
  Common approaches
- Hedging

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Realized Variance

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Realized Variance

Impact of dividends

- Assume that rates are zero and that the stock pays at dividend dates $0<\tau_1<\ldots<\tau_N$ cash dividends Δ_i. Accross a dividend date we have

$$S_{\tau_i} = S_{\tau_i -} - \Delta_i$$

- Bermudez et al (2006): since the stock price can never be below the PV of any certain future cash dividends, we can write S in terms of a martingale X as

$$S_t = \underbrace{\left(S_0 - \sum_i \Delta_i \right)}_{S_0^*} X_t + \sum_{k:t_k>t} \Delta_k$$

Adjusted "volatile" spot level.

"Pure martingale" part

Stock floor due to future dividends

– Extension to deterministic rates, proportional dividends and credit risk straight forward, Bermudez et al (2006).

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Realized Variance

Impact of dividends

- For realized variance, we obtain

$$
\begin{aligned}
RV(T_1,T_2) &= RV_{cont}(T_1,T_2) + \sum_{k:T_1<\tau_k\le T_2} \log^2 \frac{S_{\tau_i}}{S_{\tau_i -}} \\
&= RV_{cont}(T_1,T_2) + \sum_{k:T_1<\tau_k\le T_2} \log^2 \frac{S_{\tau_i}}{S_{\tau_i}+\Delta_i}
\end{aligned}
$$

Realized variance of the "continous part"
Note that this is *not* the same as the "propoprtional dividend" RV.

– There is additional variance arising from the dividends.

– For single stocks, dividends are usually taken out.

– What happens to variance swaps ...?

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Realized Variance

Impact of dividends on variance swaps

- Assume that the stock price is continous between dividends, e.g.

$$\frac{dX_t}{X_t} = \sqrt{v_t}\, dB_t$$

- Then,

Log-contract

Continuous part of RV

$$d \log S_t \approx \frac{1}{S_{t-}} dS_t - \frac{1}{2} dRV_{cont} + \log \frac{S_t}{S_t + \Delta_t}$$

Delta-Hedge

Additional contribution from Dividends

  - Variance swap pricing needs to take into account dividends, too.
  - Note that delta-hedging strategy still works very well.

Realized Variance

A note on default risk

- Variance swaps on single stocks incorporate a „cap" on realized variance. Hence their payoff is

$$\min\left\{RV(T_1,T_2), 250\% V_0(T_1,T_2)\right\} - V_0(T_1,T_2)$$

  – The value of this cap is zero in a diffusion framework.

  – To incorporate default risk, a simple way is to say

$$(RV(T_1,T_2) - V_0(T_1,T_2))1_{\text{no default}}$$

$$+$$

$$(250\% - 1)V_0(T_1,T_2)1_{\text{default}}$$

  – Can be implemented very easily.

  – But mind effect on pricing via log-contract.

Realized Variance

Relation to quadratic variation

- We know that realised variance is an unbiased estimator of the *quadratic variation* of the returns of S,

$$QV(T_1,T_2) := \left\langle \log S_{T_2} \right\rangle - \left\langle \log S_{T_1} \right\rangle$$

- However, care must be taken if a non-linear payoff on realized variance is priced.
  - See also Barndorff-Nielssen et al (2005).

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Realized Variance

Relation to quadratic variation

– Jordinson (2006): assume a Black & Scholes model with constant vol σ.

- Then,

$$RV(0,T) = \sum_{i=1}^{n} \frac{\sigma^2 T}{n} \left(x_i - \frac{1}{2} \frac{\sigma^2 T}{n} \right)^2 = \sigma^2 T \frac{1}{n} \sum_{i=1}^{n} \left(x_i - \frac{1}{2} \frac{\sigma^2 T}{n} \right)^2$$

where $(x_i)_i$ are independent standardnormal. Hence,

$$\frac{RV(0,T)}{QV(0,T)} \sim \chi^2(n,\lambda) \qquad \lambda = \sigma^4 T^2 / n$$

- Hence, the price of an option on realized variance is non-trivial even in BS.
- Above approximation allows direct pricing in BS.

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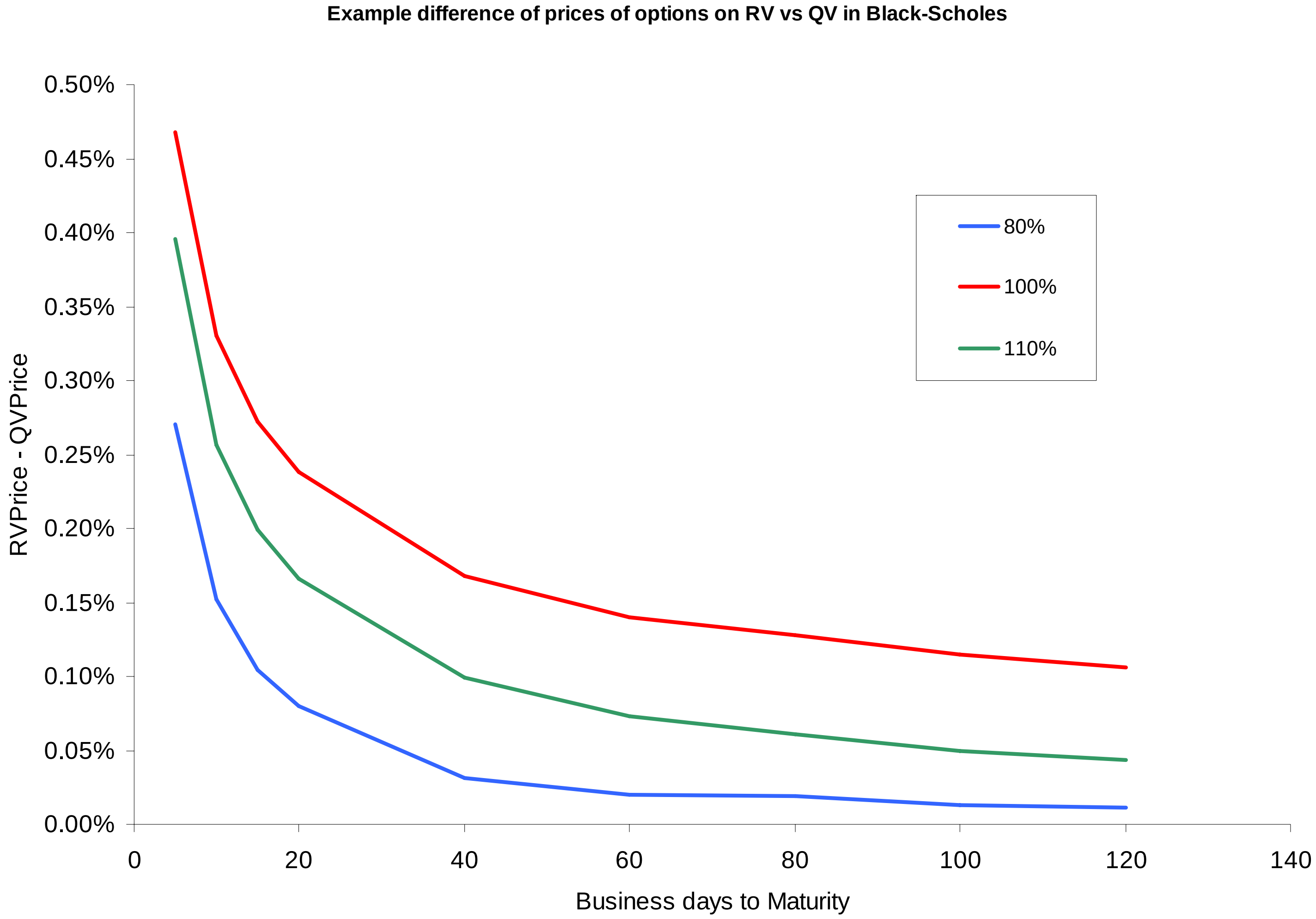
Realized Variance
Relation to quadratic variation



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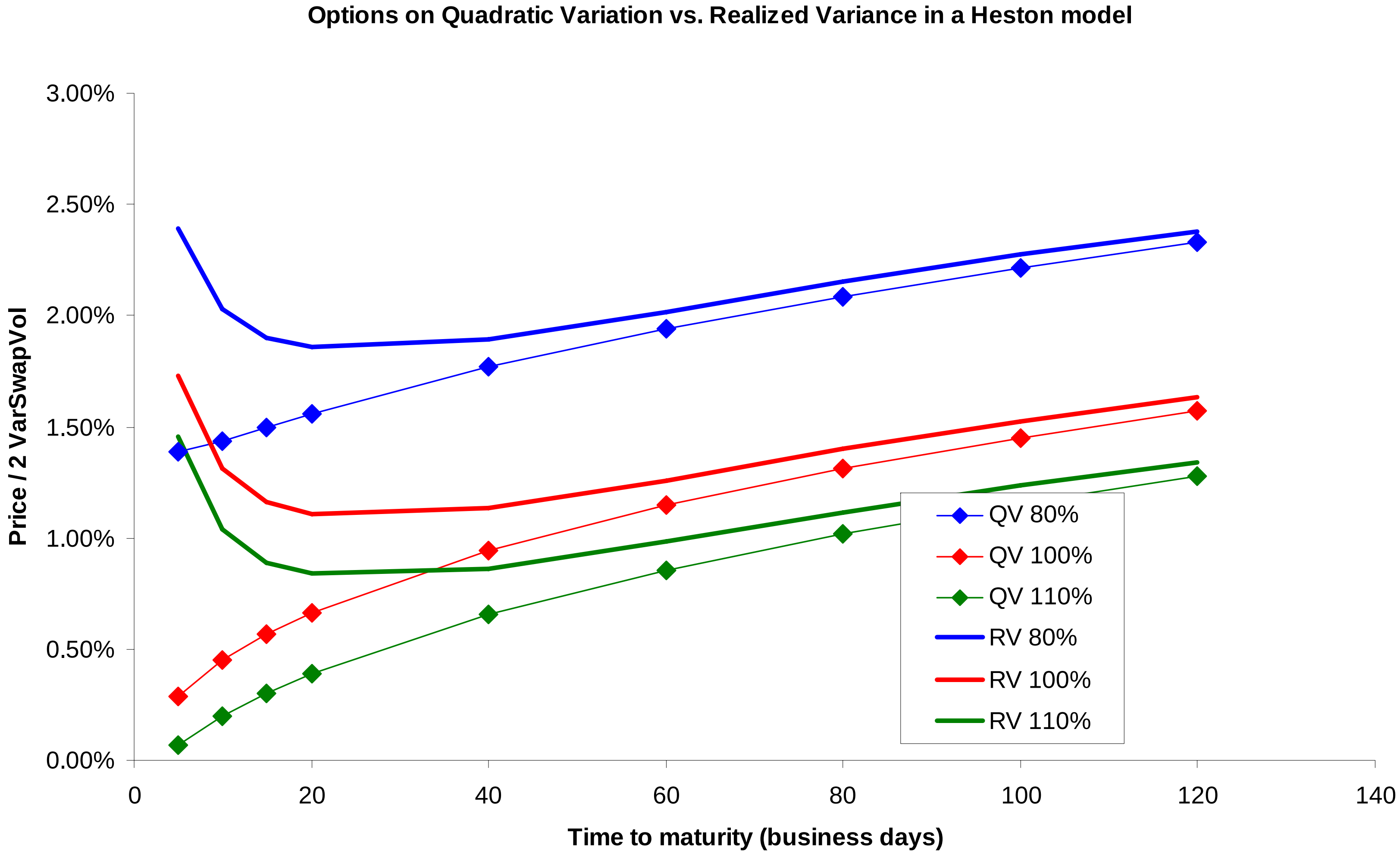
Realized Variance

Relation to quadratic variation



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Modelling Realized Variance

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Quadratic Variation
Modeling

- Despite the previous finding, it is common practise to model QV instead of RV.
  - As long as we use RV to *price* options on variance, no contradiction.

- Modelling QV - various approaches:
  - Direct modelling of the distribution of QV.
  - Consistent variance swap HJM-type term-structure models.
  - Hybrid term-structure modelling with specific stock price distribution
  - “Fitted” models

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Quadratic Variation
Direct Models

- Use the marginal distribution F_T of $QV(0,T)$ directly.

$$RV(0,T) \approx QV(0,T) \sim \text{distribution } F_T$$

- Current approaches to infer the distribution of QV rely on a symmetric smile, in which case we have

$$\text{Call}(T,K) = \int_0^\infty \text{BSCall}(K,V)\,dF_T(V)$$

  – Solve directly via numerical inversion of integral equation on discrete grid.
  – Carr/Lee (2004) use power payoffs to recover F_T.

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Quadratic Variation

Direct Models

- Good
  - No input other than market information required.
  - Pricing vanilla options simple and very quick.
  - Greeks w.r.t. variance swaps straight forward.
- But
  - Publicly known methods rely on symmetric smile (FX etc)
  - $RV \neq QV$
  - Maturity dependent; no link between different maturities per se.

- Note: information encoded in European option prices equivalent to Dupire's local volatility $\rightarrow$ base model for lower bounds for options on variance (that's a conjecture!).

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Quadratic Variation

Consistent Variance Curve Models

- HJM-type “consistent term structure models” Buehler (2006)
  - Based on consistent HJM interest rate theory by Bjoerk/Filipovic/Teichman and others.
  - Attempt to model the full term structure movements of variance swaps and to use the latter as natural hedging instruments.

- Notation:
  - A variance swap price (ex scaling) is denoted by

$$V_t(T_1,T_2) := \mathrm{E}_t\left[RV(T_1,T_2)\right]$$

  - A more convenient object is the “floating maturity” variance swap

$$U_t(x) := V_t(t,t+x)$$

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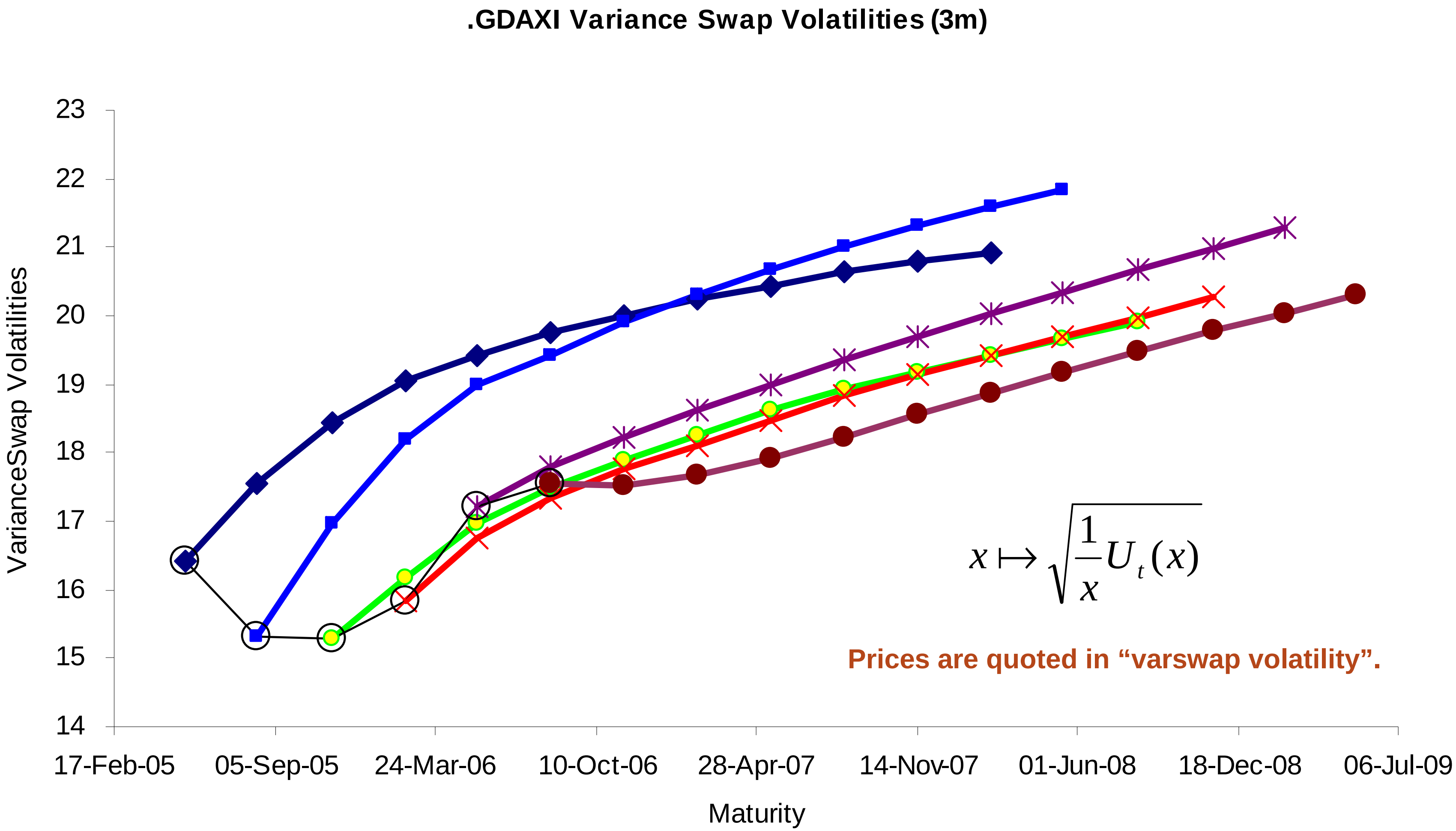
Quadratic Variation

Consistent Variance Curve Models



$$x \mapsto \sqrt{\frac{1}{x} U_t(x)}$$

Prices are quoted in “varswap volatility”.

Quadratic Variation

Consistent Variance Curve Models

- "Consistency" means that the variance curves U_t belong to some finite-dimensional function class ("finite dimensional representation").
  - This ensures controlled term-structure movements.
- Hence, write $U(x)$ as a function of a low-dimensional parameter process Z.
  - Since U as a function of x must be increasing, it is easier to write its derivative as non-negative function G of Z, i.e.

$$U_t(x) \equiv \int_0^x G(y; \vec{Z}_t)\,dy$$

G is the "variance curve function" and describes a "typical" shape of the implied variance swap curve.

with

$$d\vec{Z}_t = \mu(\vec{Z}_t)dt + \sum_{j=1}^{d} \sigma_j(\vec{Z}_t)d\vec{W}_t^j$$

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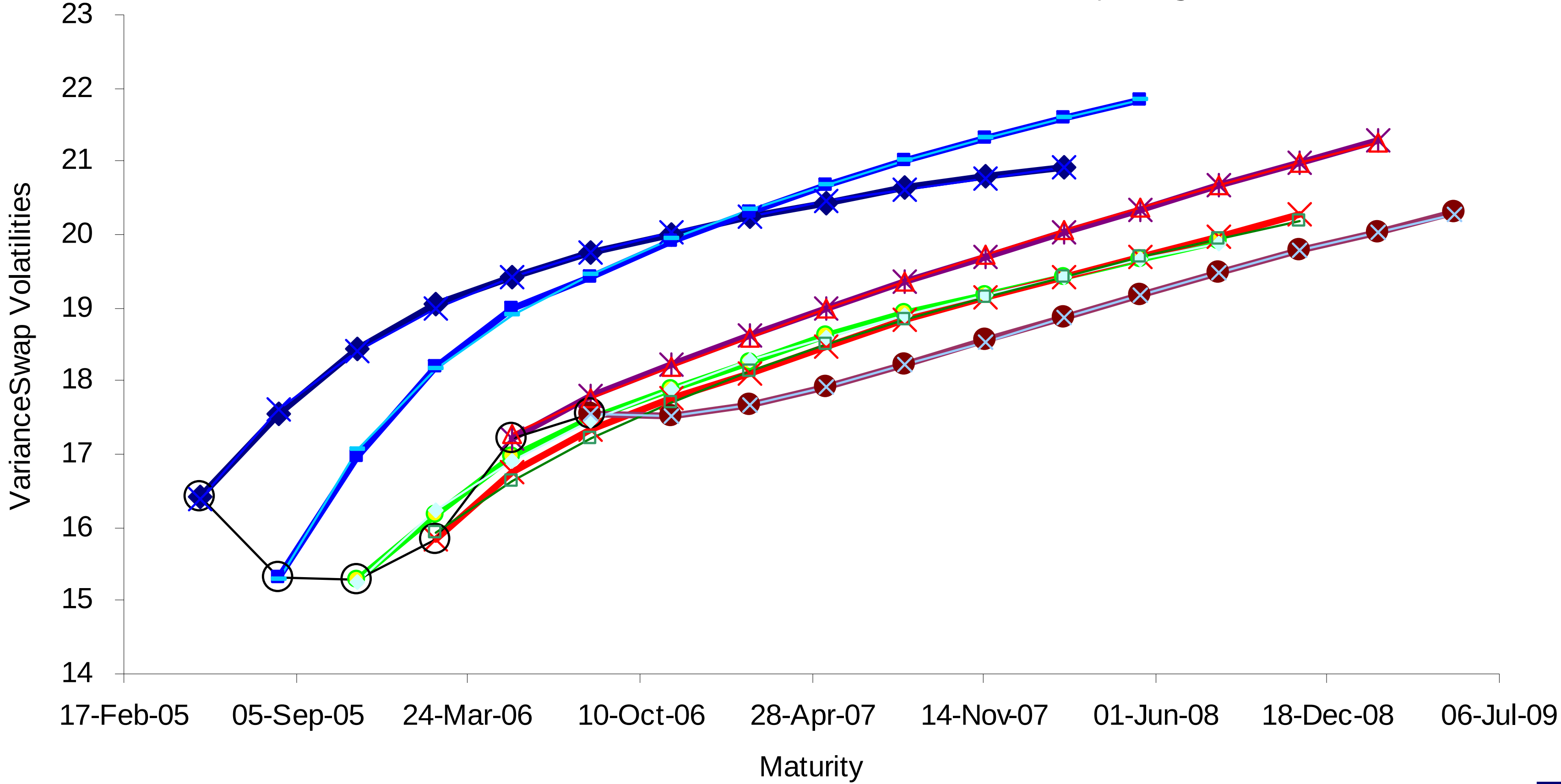
Quadratic Variation

Consistent Variance Curve Models

$$G(x; z_1, \ldots, z_5) := z_3 + (z_1 - z_3)e^{-z_4 x} + (z_2 - z_3)\frac{z_4}{z_4 - z_5}\left(e^{-z_5 x} - e^{-z_4 x}\right)$$



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Quadratic Variation
Consistent Variance Curve Models

- The (local) martingale property of the variance swap prices imposes restrictions on the possible coefficients μ and σ of the SDE of Z.

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^T \sigma(z)\,\partial^2_{zz} G(z;x)$$

- If satisfied, we can always define

Short variance implied by this approach.

$$\frac{dS_t}{S_t} = \sum_{j=1}^{d} \rho^j (S_t, \vec{Z}_t)\left(\sqrt{v_t}\, dW_t^j\right) \qquad v_t := G(0;\vec{Z}_t)$$

where ρ with $\|\rho\|_2 = 1$ is a local correlation vector which controls the skew of the implied volatility.

– Arbitrage-free model

– Complete under differentiability assumptions on the coefficients of the SDEs.

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Quadratic Variation

Consistent Variance Curve Models

- An example (see Bermudez et al 2006) is

$$G(x; z_1, \ldots, z_3) := z_3 + (z_1 - z_3)e^{-kx} + (z_2 - z_3)\frac{k}{k-c}\left(e^{-cx} - e^{-kx}\right)$$

with driving Heston-type process

$$\begin{aligned} dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \nu\sqrt{Z_t^1}dW_t^1 \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \mu\sqrt{Z_t^2}dW_t^2 \\ dZ_t^3 &= const \end{aligned}$$

– If W^1 and W^2 are independent, Fourier-transform can be computed using Ricatti equations.

– More elaborate volatility structure requires numerical calibration via MC.

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Consistent Variance Curve Models

- Good
  - Fully consistent arbitrage-free joint stock/variance “stochastic volatility” model
  - Greeks w.r.t. stock and variance swaps straight forward.
  - Consistent term structure dynamics.
  - Self-similar future implied volatility surfaces by construction.
- But
  - “Skew” realized via correlation coefficient ρ; very indirect.
  - Cumbersome non-linear calibration for most models except for affine sub-class.

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Quadratic Variation

Hybrid models

- Hybrid models – model term structure of variance swaps but use mathematically consistent “locally controlled” stock dynamics for intervalls between “reset dates” $t_0 < \cdots < t_N$:
  - Assume the “short variance” v is given.
  - On $(t_{k-1}, t_k]$ set

Stochastic or local volatility

$$S_t := S_{t_{k-1}} X_t^k \qquad \frac{dX_t^k}{X_t^k} = \sigma_t^k\left(X_t^k\right) dB_t \qquad \left(X_{t_{k-1}}^k = 1\right)$$

such that

$$\mathrm{E}_{t_{k-1}}\left[RV(t_{k-1}, t_k)\right] = \mathrm{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} \sigma_t^k\left(X_t^k\right)^2 dt\right]$$

$$\equiv$$

$$V_{t_{k-1}}(t_{k-1}, t_k) := \mathrm{E}_{t_{k-1}}\left[\int_{t_{k-1}}^{t_k} v_s \, ds\right]$$

Note that realized variance is computed using X^k.

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Quadratic Variation
Hybrid models

- Examples:
  - Bergomi 2005: two-factor exponential-OU model for v and CEV distribution for the stock price increments between reset dates.
    - Model intended to hedge Cliquets using (forward) variance swaps
    - Approximation formulas can be used to “fix the smile” for the forward intervals.
    - No closed form for spot Europeans, but good “natural fit”.
  - Bermudez et al 2006: one-factor Heston model with Heston forward distribution
    - Pricing tool for assessing cross-correlation risk when pricing Cliquets.
    - Allows to control inter-dependency between the increments in Heston.
    - Analytic Fourier-transform available to calibrate spot European options.
    - Forward distribution simply Heston.

Quadratic Variation

Hybrid models

- Good
  - Mathematically consistent
  - Greeks w.r.t. stock and variance swaps straight forward.
  - Consistent term structure dynamics.
  - Self-similar future implied volatility surfaces at each reset date by construction.
- But
  - Logically inconsistent short variance behaviour across reset dates.
  - Reset-date dependent.
  - Spot implied vol surface more or less random if models are calibrated towards the forward distribution.

Quadratic Variation

Fitted models

- Dupire 2004: use log-normal variance and fit it to the observed market prices of variance swaps.
  - Assume that at time 0 we observe a variance curve

$$V_0(0,T) = \int_0^T w(s)ds$$

  - Define the drift μ in short-variance process

$$\frac{dv_t}{v_t} = \mu_t dt + \sigma_t dW_t$$

  such that

$$V_0(0,T) = \mathrm{E}\left[\int_0^T v_s ds\right]$$

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Quadratic Variation

Fitted models

- Such a model is always perfectly fitted to the variance swap market. More general, we can extend this "Hull&White-approach" to any short variance model $(x_t)_t$ (Buehler 2006):
  - Given x, rescale

$$v_t := w(t)\frac{x_t}{\mathrm{E}[x_t]} \qquad V_0(0,T) = \int_0^T w(s)ds$$

such that, obviously,

$$V_0(0,T) = \mathrm{E}\left[\int_0^T v_s ds\right]$$

w(t) is the "forward variance" at time t.

  - To construct a stock price process, chose a BM B and set

$$\frac{dS_t}{S_t} = \sqrt{v_t}\,dB_t$$

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Quadratic Variation
Fitted models

- Examples
  - Dupire 2004: Log-normal model.
  - Bergomi 2005: Two-factor exponential-OU.
  - Buehler 2006: Heston one- and two-factor version
    - Europeans can be priced via Fourier-inversion by solving Ricatti-equations.
    - Analytic Fourier-transform if time-change method is used.
    - Consistent models remain consistent if correction is not “too large”.

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Quadratic Variation
Fitted models

- Good
  - Perfect fit to the hedging instrument variance swap for options at all maturities.
  - Greeks w.r.t. stock and variance swaps straight forward.
  - Consistent term structure dynamics if combined with a consistent model.
  - For Heston-based models relatively quick European prices on the stock.
- But
  - Ad hoc definition could lead to inconsistent term-structure movements.
    But by experience, models are remarkably resistent.
  - Skew incorporated only indirectly via correlation parameter.

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Jumps

Introducing skew

- The implied volatility skew for options on realized variance is upward sloping.
  - This reflects the additional risk of jumps in the stock price which always contribute to an increase of realized variance.

$$RV(T_1,T_2) \approx QV_{cont}(T_1,T_2) + \sum_{t:T_1<t\leq T_2} \log^2 \frac{S_t}{S_{t-}}$$

  - Hence, more risk premium is charged for OTM calls on variance.

- Adding jumps can improve the fit to the broker market of options on realized variance.
  - “Fitting” can easily be applied to this case.
  - Hedging arguments theoretically less clear.

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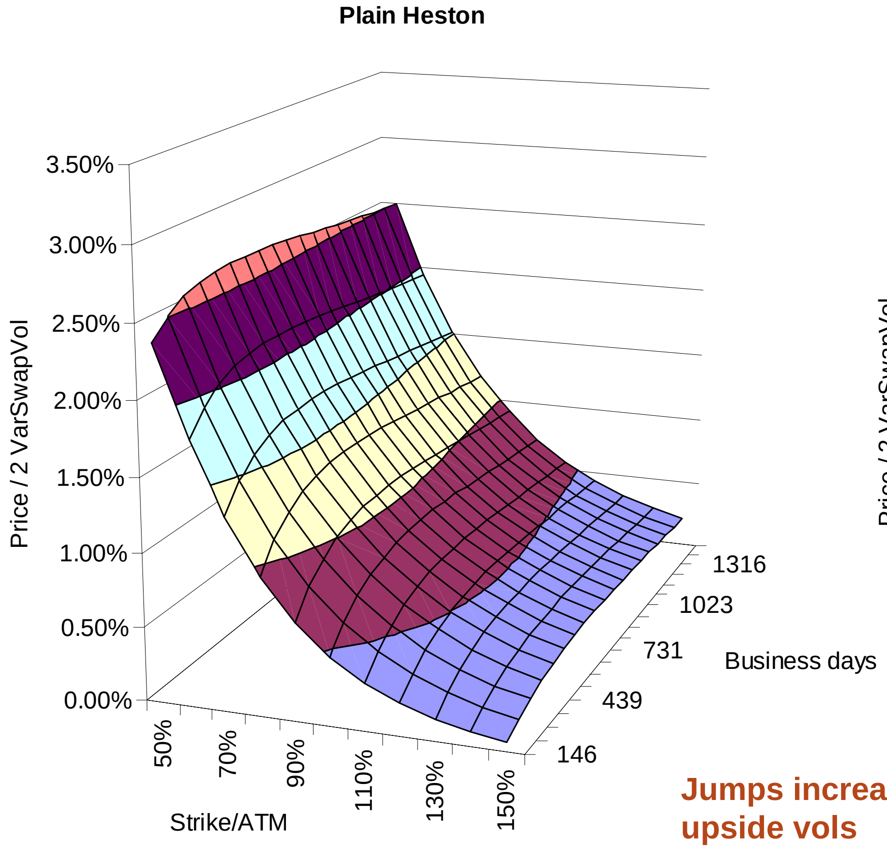
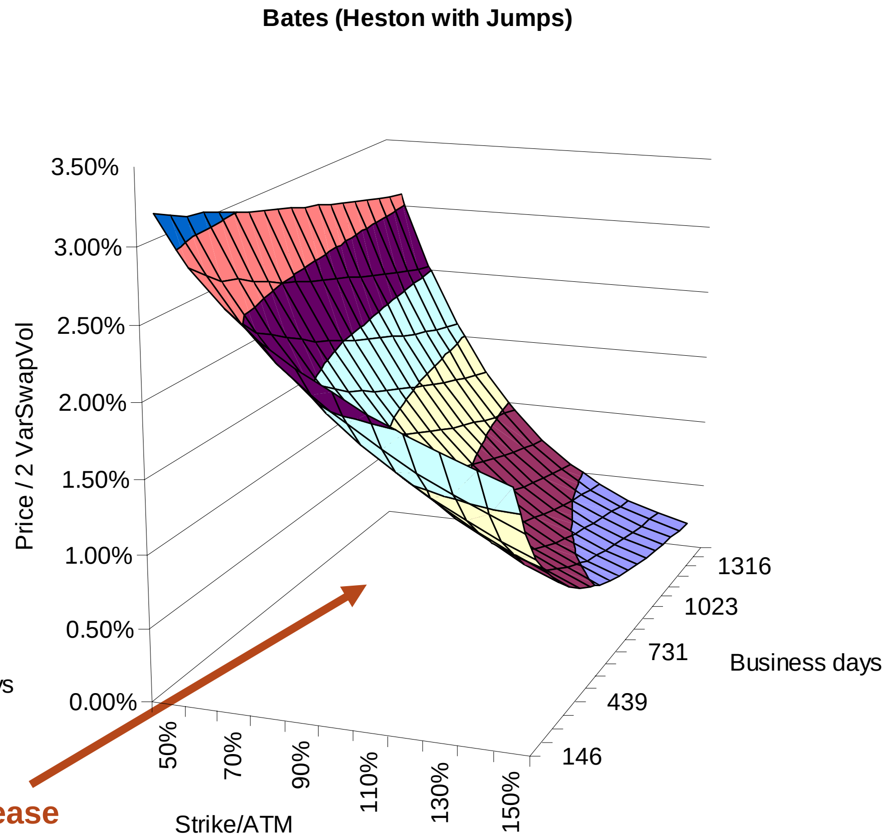
Jumps
Impact of jumps

<u>Bates model vs Heston</u>

VolOfVol in Heston calibrated to Bates equity 1yATM.
Both models have 20% flat variance swap vol.





Jumps increase upside vols (here seen in price)

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Hedging

Hedging

… with variance swaps.

- The most natural instrument to hedge an options on variance is a variance swap. Set

$$H_t(T) := \mathrm{E}_t\left[h\left(RV(0,T)\right)\right]$$

- Most models we described provide us with a delta with respect to both stock price and the variance swap with the same maturity as the option.
- In theory, for any one-factor SV model, we have

$$dH_t(T) = \Delta_t dS_t + \Theta_t dV_t(0,T)$$

Stock-delta

VarSwap-Delta.

Hedging

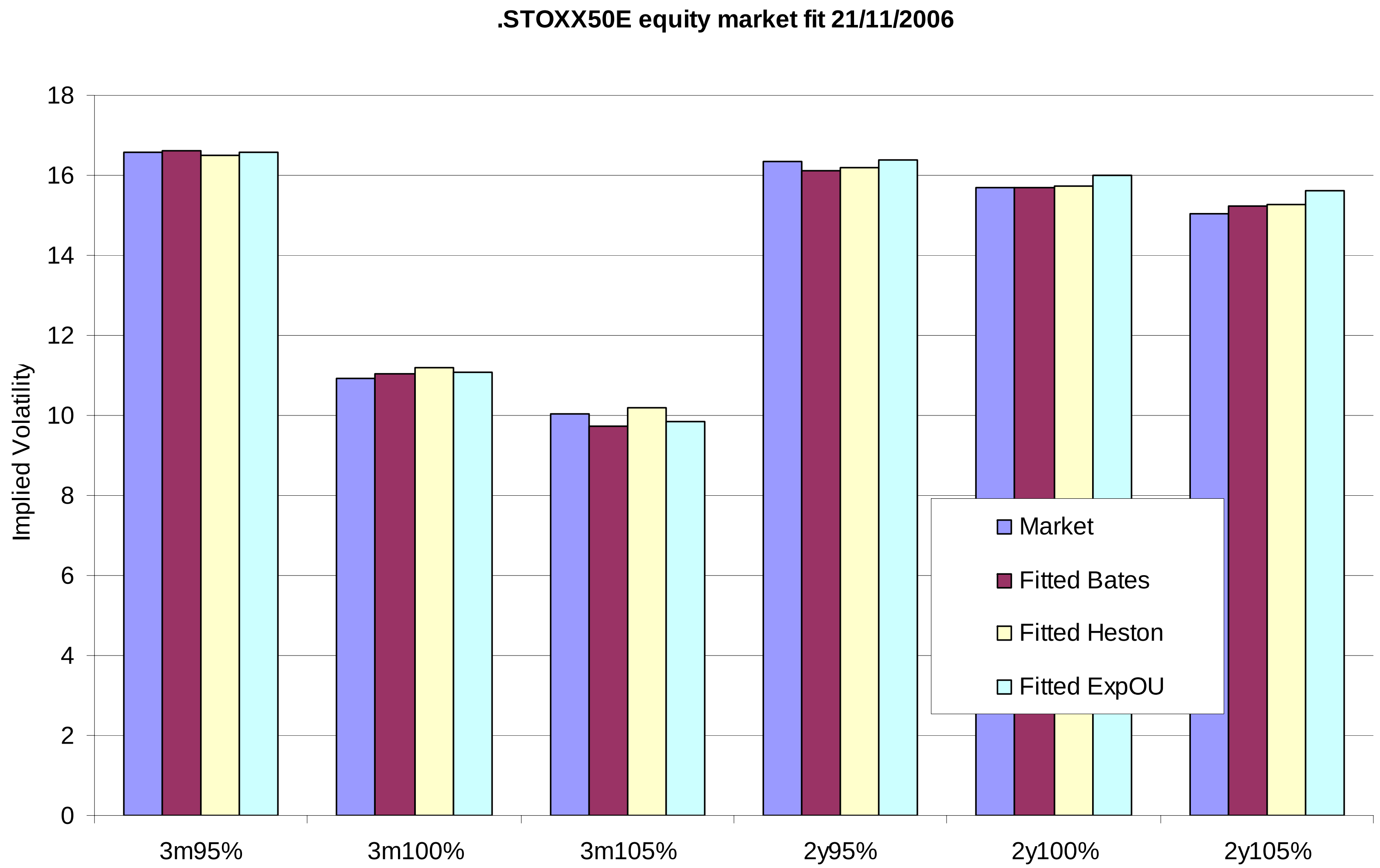
… with variance swaps.

- We conduct a test:
  calibrate three models ATM and ±5% OTM options on spot
  - Fitted Heston
  - Fitted Bates (Heston with Jumps)
  - Fitted ExpOU with jumps

- We want to see
  - Impact of the above on price and variance swap delta reported by the model.

Hedging

… with variance swaps.



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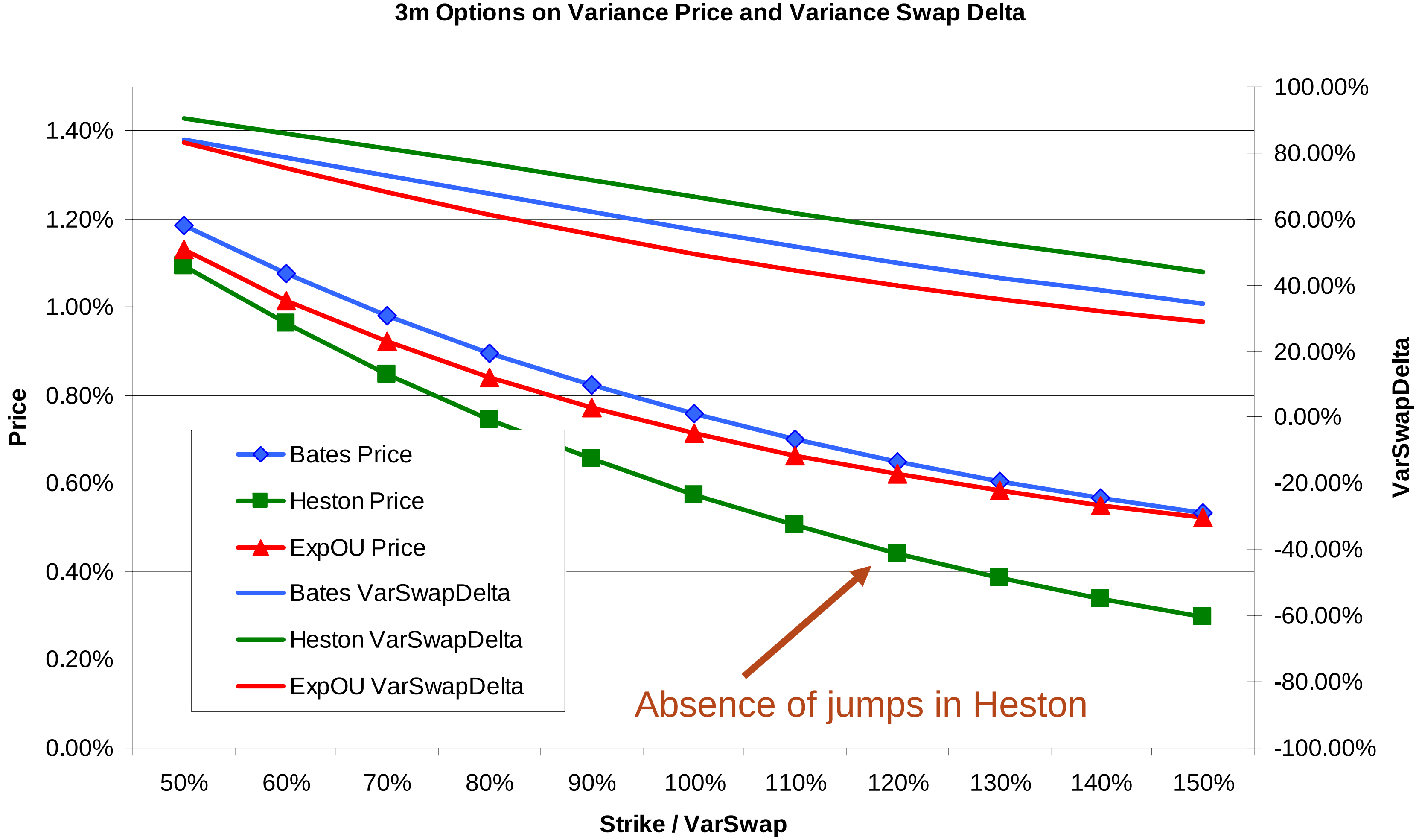
Hedging

… with variance swaps.



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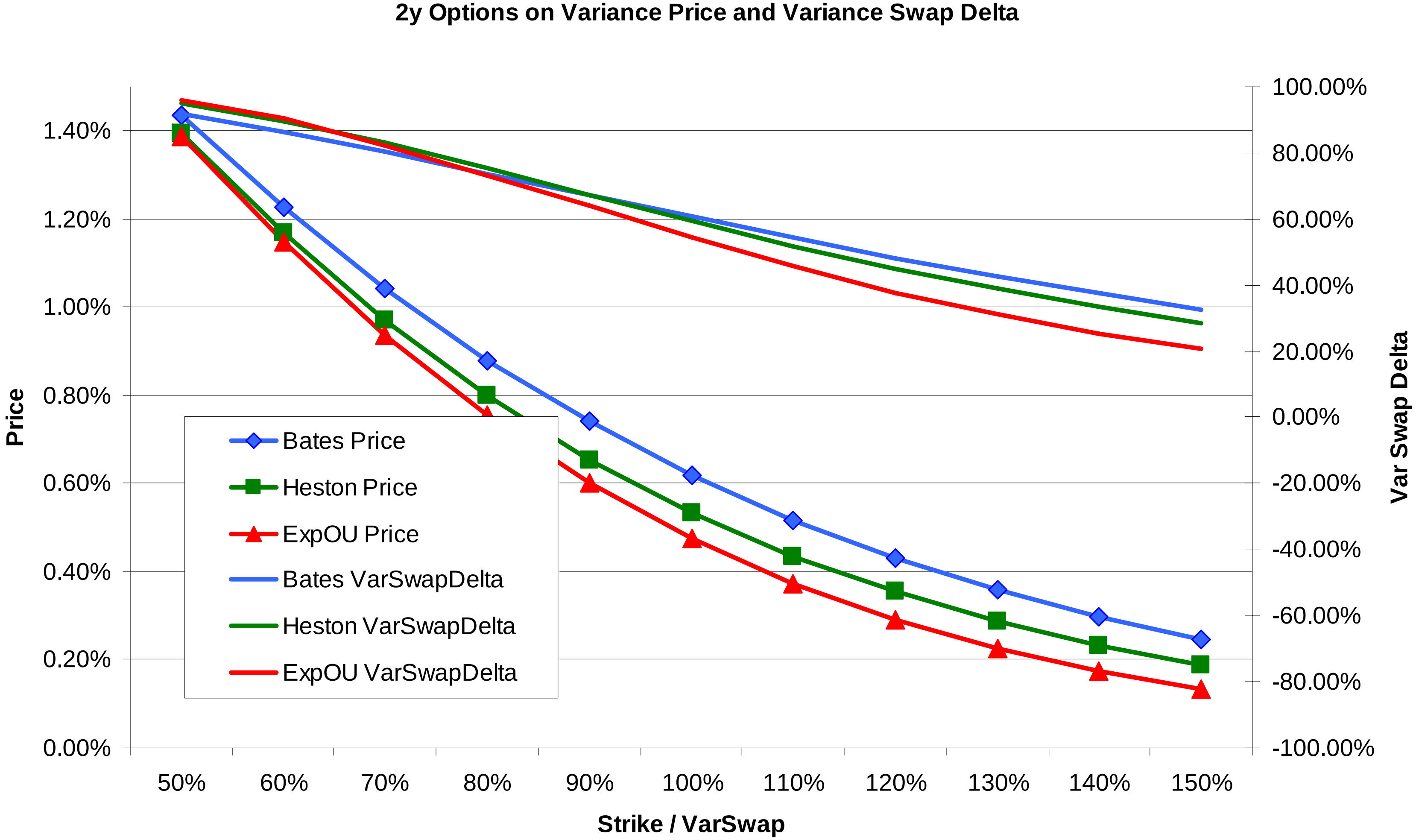
Hedging

… with variance swaps.



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Pricing and Hedging in Practise

… more points to look out for

- The idea of hedging with variance swaps is nice, but in practise we face model risk and substantial hedging costs.
  - Approximate variance swap log-profile by far less vanillas and use these to hedge your position.
  - Adapt vol-of-vol accordingly.
  - Diversify

Thank you very much for your attention.

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Quadratic Variation
Fitted models

- This is a close relative to time-changing the process
  - Let

$$A(t) := Y^{-1}(V_0(0,T)) \qquad Y(t) := \mathrm{E}\left[\int_0^t x_s \, ds\right]$$

Y(t) is the variance swap price given by the model *x*.

and set

$$S_t := X_{A(t)} \qquad \frac{dX_t}{X_t} = \sqrt{x_t}\, dB_t$$

such that once again

$$V_0(0,T) = \mathrm{E}\left[QV(0,T)\right]$$

  - If we have a closed form for the price of spot European options in the model *(X,x)*, it is very easy to compute the price for *(S,v)* simply by adjusting the maturity of the option accordingly.