Hedging Options On Variance

Hans Buehler
Deutsche Bank GME Quantitative Products Analytics, London

hans.buehler@db.com
http://www.dbquant.com

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Hedging Options On Variance

- Measuring Hedging Performance
  - Formuling the problem
- Model Candidates
- Results

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Hedging Options On Variance

Fix some notation

- *Realised variance* is defined over business days $T_1=t_0<\cdots<t_n=T_2$ as

$$RV(T_1,T_2) := \sum_{i=1}^{n}\left(\log\frac{S_{t_t}}{S_{t_{t-1}}}\right)^2$$

  – The scaled quantity $1/n\ RV(T_1,T_2)$ is an unbiased estimator for the quadratic variation of $\log S$ in $[T_1,T_2]$ ie

$$RV(T_1,T_2) \approx \mathrm{E}\left[\mathrm{QV}(T_1,T_2) := \langle \log S\rangle_{T_2} - \langle \log S\rangle_{T_1}\right]$$

  – We denote spot started variance swaps as V:

$$V_t(T) := \mathrm{E}\left[\mathrm{R}V(t,T)\,|\,F_t\right]$$

Hedging Options on Variance

Measuring hedging performance

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Options on Realized Variance

Measuring hedging performance

- The eternal model question

Which Model is the Best ?

→ The one which gives me „the best“ hedging performance

Options on Realized Variance

Measuring hedging performance – formulating the problem

- What do we mean by *hedging performance*?

1. Any bug-free and well-understood model will allow to do full P&L explanation:
   - Some degree of Taylor expansion of the model pricer in terms of all inputs will explain the changes of prices.
   - However, that does not tell us how to actually hedge ourselves.

2. Our approach here:
   Hedge options on realized variance using „variance swap delta".
   - Calibrate our models daily using historic data.
   - Price, and hedge our variance swap and stock delta.
   - Compute daily hedging error.

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Options on Realized Variance

Measuring hedging performance – some theory

- In standard stochastic-volatility models, we have

$$
\begin{aligned}
\frac{dS_t}{S_t} &= \mu_t dt + \Sigma(S_t, v_t) dW_t \\
dv_t &= a(v_t) dt + b(v_t) dB_t
\end{aligned}
$$

No jumps

- For “any” path-dependent payoff H with maturity T denote by P the price

$$P_t^T(s;v) := \mathrm{E}\left[\mathrm{H} \mid S_t = s; v_t = v; \mathrm{F}_{t-}\right]$$

  – Here, F_t is the predictable σ-algebra. Example:

$$P_t^T(s;v) := \mathrm{E}\left[\left(rv + RV(t,T)\text{-}K\right)^+ \mid S_t = s; v_t = v; RV(0,t) = rv\right]$$

Options on Realized Variance

Measuring hedging performance – some theory

- Under some weak conditions (e.g. Buehler 2006) we can write

$$dP_t^T(S_t;v_t) = \delta_t dS_t + \omega_v^T dV_t(T) + drift$$

- Define the (stock) delta $\Delta_t := \partial_S P_t^T(S_t;v_t)$

  and "VarSwapDelta"

$$\Omega_t^T := \frac{\partial_v P_t^T(S_t;v_t)}{\partial_v V_t(T)}$$

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Options on Realized Variance

Measuring hedging performance – some theory

- Note that given our discussion about realized variance and quadratic variation, the variance swap itself has some small delta Δ^V in the model (this is *not* the standard 2/S(t) delta!).

- We therefore have

$$dP_t^T(S_t;v_t) = \underbrace{\left(\Delta_t - \Omega_t^T \Delta_t^V\right)}_{\tilde{\Delta}_t} dS_t + \Omega_v^T dV_t(T)$$

- Then, define the *instantaneous hedging error* as

$$\varepsilon_t^T := \frac{\tilde{\Delta}_t dS_t + \Omega_t^T dV_t(T) - dP_t^T(S_t;v_t)}{P_t^T(S_t;v_t)}$$

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Options on Realized Variance

Measuring hedging performance – formulating the problem

- The *instantaneous hedging error* is the relative short-fall of our hedged position with respect to the value of the option.

$$\varepsilon_t^T := \frac{\tilde{\Delta}_t dS_t + \Omega_t^T dV_t(T) - dP_t^T(S_t; v_t)}{P_t^T(S_t; v_t)}$$

  – Note that we actually going to re-calibrate the model between t and $t+1$, hence interpret the above accordingly.

  – The definition makes only sense for payoffs $H>0$ ($\rightarrow$ separate swap legs).

Options on Realized Variance

Measuring hedging performance – formulating the problem

- We want a model such that ε is "small"

$$\varepsilon_t^T := \frac{\tilde{\Delta}_t dS_t + \Omega_t^T dV_t(T) - dP_t^T(S_t; v_t)}{P_t^T(S_t; v_t)}$$

  - What does this mean?
  - The instantaneous error is a random variable, we observe only one path.

- *The primary aim is to reduce the variance of the relative daily changes of the position, i.e. "how much uncertainty" there is.*

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Options on Realized Variance

Measuring hedging performance – formulating the problem

- Assume we want to know "the error" for 3m ATM calls on RV.

1. Fixed maturity view:
   - Slice the available data in consecutive 3m regions
   - Compute OOV hedges and payoffs for all relevant strikes.
   - Compute for each the average daily hedging error and its standard deviation.

$$e := \frac{1}{n}\sum_k \varepsilon_{t_k}^{T_i} \qquad \sigma := \sqrt{\frac{1}{n(n-1)}\sum_k \left(\varepsilon_{t_k}^{T_i} - e\right)^2}$$

   - Problem:
     - Data very uninteresting when option moves out-of- or in-the-money (in both cases the hedges obviously work very well).
     - Only very limited amount of data does not allow many samples of the above.

Options on Realized Variance

Measuring hedging performance – formulating the problem

2. Floating maturity view:
   - Daily, compute the daily hedging error for calls which started yesterday.
   - This allows to concentrate on the region with the highest VarSwapGamma.
   - Much more relevant data can be obtained this way.

$$e := \frac{1}{n}\sum_k \varepsilon_{t_k}^{t_k+3m} \qquad \sigma := \sqrt{\frac{1}{n(n-1)}\sum_k \left(\varepsilon_{t_k}^{t_k+3m} - e\right)^2}$$

- We then compute the mean/stddev for various maturities to get an idea where the hedging performance is most critical.

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Options on Realized Variance
Measuring hedging performance - calibration

- All the models we use will have some “VolOfVol” parameter.
  - This parameter will be calibrated every day and used to price the option
    - Use ATM European option on the equity to extract VolOfVol as a measure of distance between VarianceSwap strike and Implied ATM Volatility.
      (cf. Buehler 2006)
  - Correlation gets fixed at -70%
  - With the above, we can also calibrate numerically expensive models.

Options on Realized Variance

Measuring hedging performance - summary

- For each model
  - We perform daily calibration of VolOfVol from equity ATM options.
  - We use VarSwapDelta and stock delta from the model to hedge the target option.
  - We look at daily relative hedging errors for the “floating” option (Mean and standard deviation)

- What we will not do
  - Impact of jumps
  - Transaction costs & the term structure of OOVs

Options on Realized Variance

Measuring hedging performance - dry run

- First a dry run:

  Asian 1m ATM Call on STOXX50E *<u>on the equity</u>*

$$H := \left(\frac{1}{n} \sum_{i=1}^{1m} S_{t_i} - F_{1m} \right)^+$$

  - Model: Black-Scholes with Moment-matching
  - Hedging instrument: Zero-strike Asian Call STOXX50E
  - Vega-Hedging using the ATM European option

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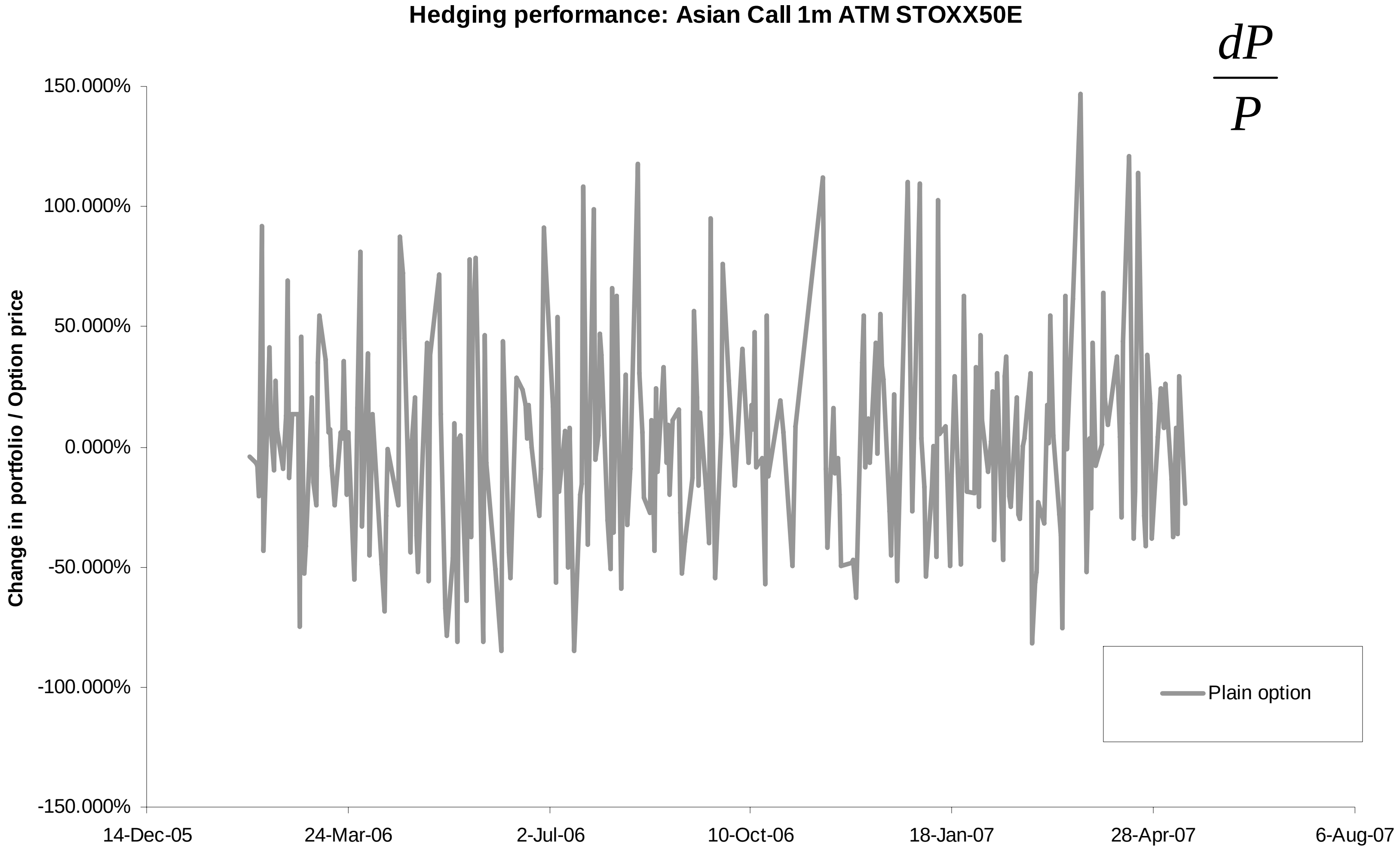
Options on Realized Variance

Measuring hedging performance - dry run



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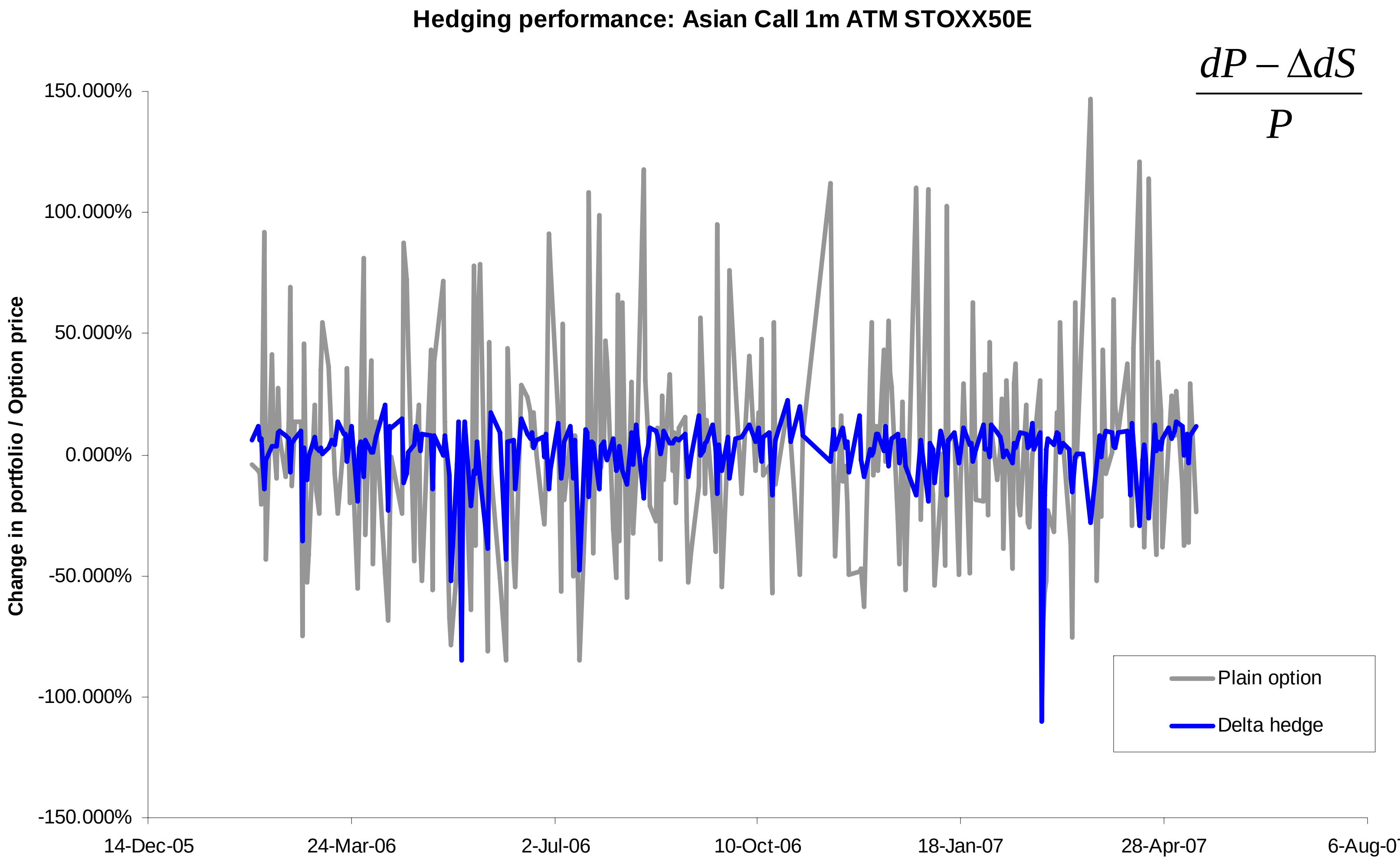
Options on Realized Variance

Measuring hedging performance - dry run



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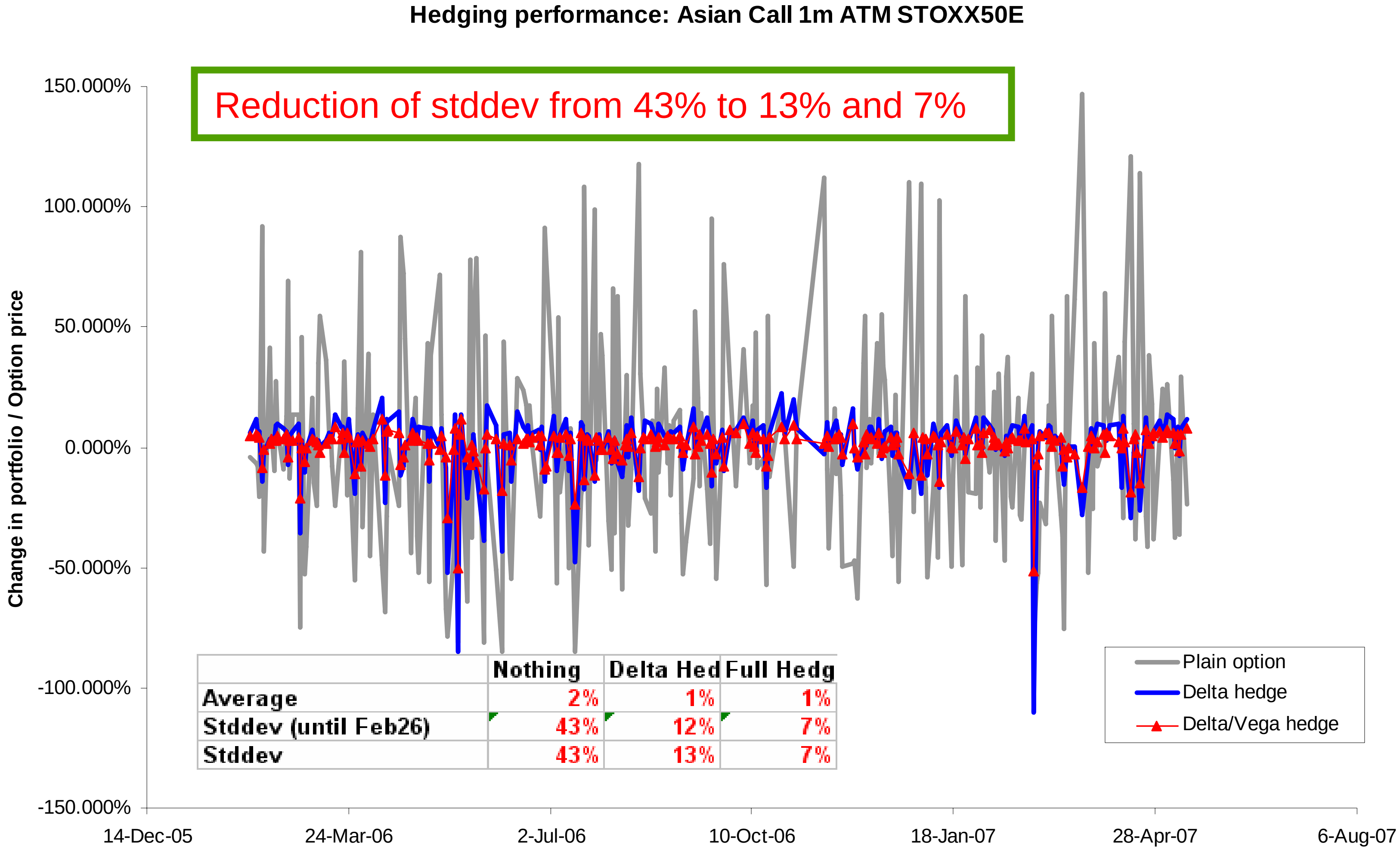
Options on Realized Variance

Measuring hedging performance - dry run



	Nothing	Delta Hed	Full Hedg
Average	2%	1%	1%
Stddev (until Feb26)	43%	12%	7%
Stddev	43%	13%	7%

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Hedging Options on Variance

Model candidates

Options on Realized Variance

Candidate models: Black Scholes RV

- *Black-Scholes for realized variance*

$$QV(T) = V_0 \cdot \exp\left\{\sigma\sqrt{T} \cdot Y - \tfrac{1}{2}\sigma^2 T\right\}$$

VolOfVol

for Y normal.

- Use usual European pricer to evaluate option.
- Calibration:
  1. Estimate σ from historic data ~ 180% for 1m
  2. For an independent Z, let

$$S_T = F_T \exp\left\{\sqrt{QV(T)} \cdot Z - \tfrac{1}{2} QV(T)\right\}$$

and obtain VolOfVol from ATM European option on the equity.

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Options on Realized Variance

Candidate models: Black Scholes RV

- What is good about it
  - Simple & easy & fast
  - Can use Black-Scholes pricer
- More properties
  - No skew (if correlation is not zero, S not a martingale)
  - No (similar) “instantaneous” equivalent which is consistent across maturities.

Options on Realized Variance

Candidate models: Fitted Heston

- *Fitted Heston* (cf. Buehler 2006, Bermudez et al 2006)
  - Use Heston dynamics and retro-fit the observed variance swap curve $V(T)$ to it. (just as Vasicek's model is fitted to the bond market)

$$
\begin{aligned}
\frac{dX_t}{X_t} &= \sqrt{v_t}\, dB_t \\
v_t &:= \frac{\tilde{v}_t}{\mathrm{E}\tilde{v}_t} \partial_T \big|_{T=t} \left(V_0(T) \right) \\
d\tilde{v}_t &= k(\tilde{v}_0 - \tilde{v}_t)dt + \sigma \sqrt{\tilde{v}_t}\, dW_t
\end{aligned}
$$

VolOfVol

  - Then we get

$$
\mathrm{E}\left[\mathrm{RV}(0,\mathrm{T})\right] = \int_0^T \mathrm{E}\left[v_t\right] dt = V_0(T)
$$

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Options on Realized Variance

Candidate models: Fitted Heston

- What is good about it
  - European option pricing very efficient, so VolOfVol calibration very fast.
  - “Proper model” with consistent dynamics across maturities
    - Akin to BS for equity, a term-structure of VolOfVol used here, though.

- More properties
  - Reversion Speed κ controls VolOfVol term-structure
  - We can add jumps to generate upside smile observed in OOV market.
  - Variance swap dynamics also affine.

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Options on Realized Variance

Candidate models: Fitted Log-Normal

- *Fitted Log-Normal* (cf. Dupire 2004, Bergomi 2005)
  - Use log-normal variance and retro-fit the variance swaps.

$$
\begin{aligned}
\frac{dX_t}{X_t} &= \sqrt{v_t}\, dB_t \\
v_t &:= \frac{e^{u_t}}{\mathrm{E} e^{u_t}} \partial_T \big|_{T=t} \left(V_0(T) \right) \\
du_t &= k(u_0 - u_t)\,dt + \sigma dW_t
\end{aligned}
$$

VolOfVol

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Options on Realized Variance

Candidate models: Fitted Log-Normal

- What is good about it
  - Very intuitive model (“canonical” approach)
  - Again - “proper model” with consistent dynamics across maturities
    - Akin to BS for equity, a term-structure of VolOfVol used here, though.

- More properties
  - No analytic European option pricer (but fairly efficient MC possible)
  - Reversion Speed κ controls VolOfVol term-structure
  - We can add jumps to generate upside smile observed in OOV market.
  - If $V(T)$ is flat, variance swaps are log-normal.

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Hedging Options on Variance

In Action

Options on Realized Variance

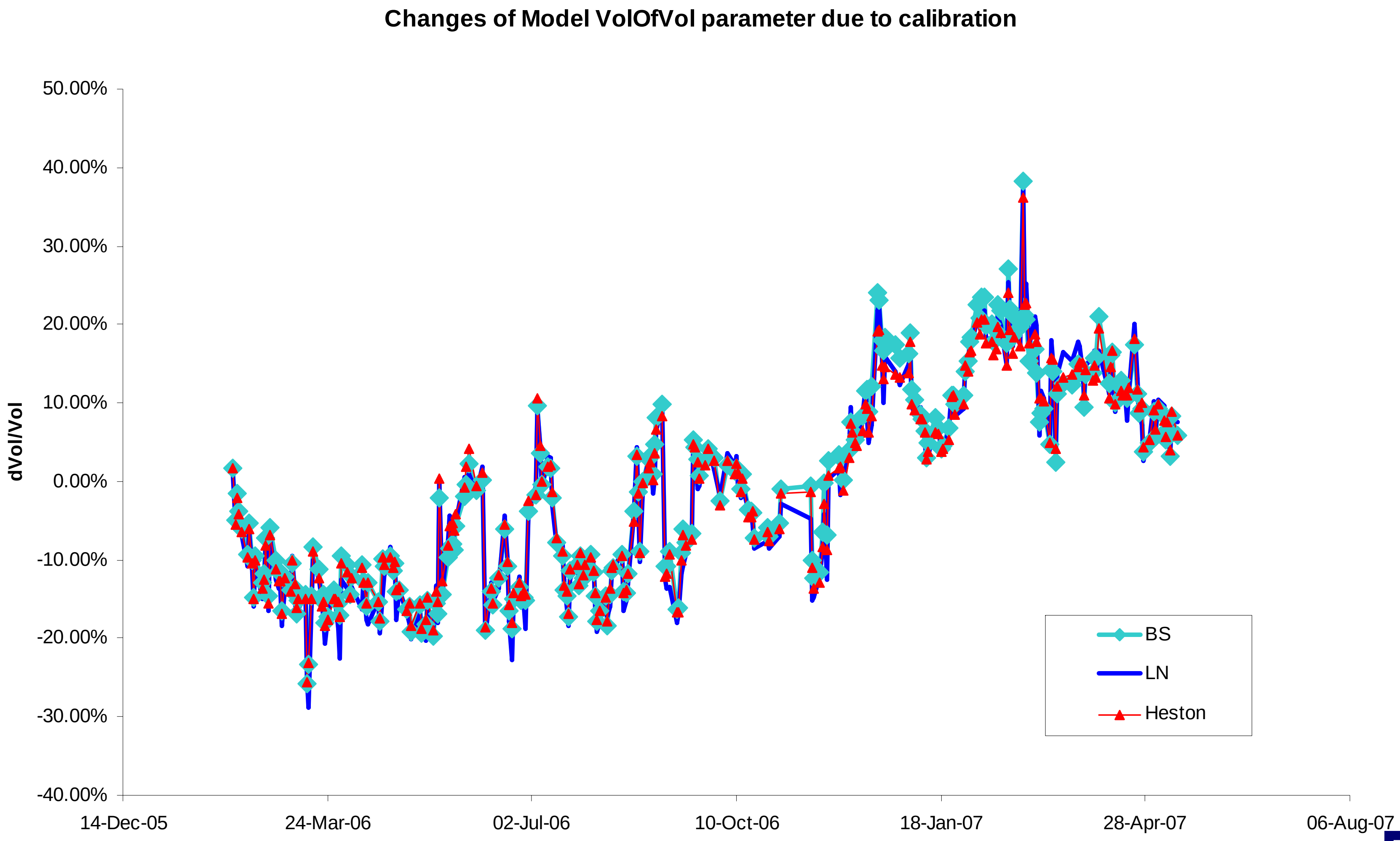
In Action

We start with a 1m ATM realized variance call

- LogNormal and Heston Reversion Speed 5

Options on Realized Variance

Measuring hedging performance



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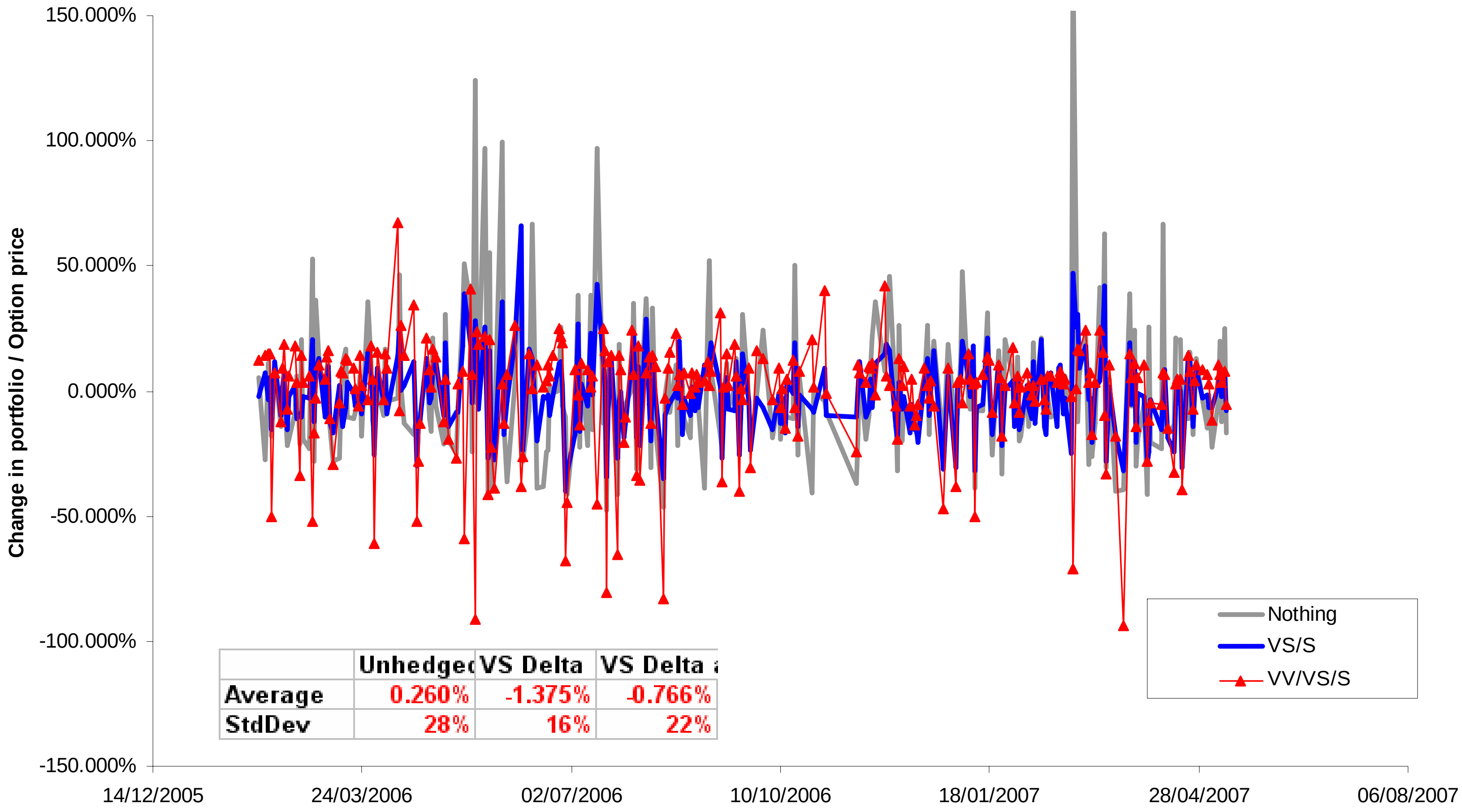
Options on Realized Variance

Measuring hedging performance

Hedging performance: OOV 1m Call, BS calibrated



	Unhedged	VS Delta	VS Delta a
Average	0.260%	-1.375%	-0.766%
StdDev	28%	16%	22%

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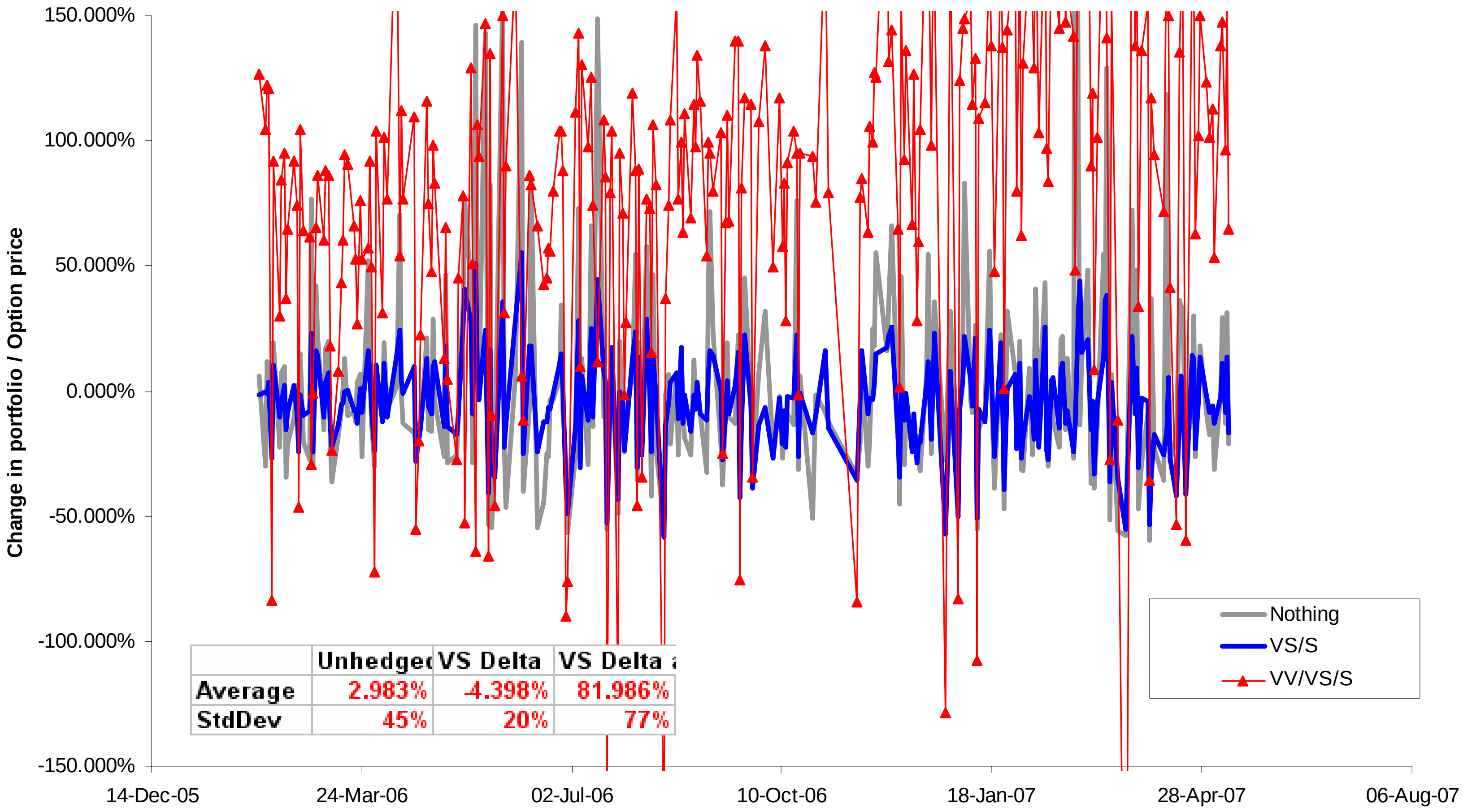
Options on Realized Variance

Measuring hedging performance

Hedging performance: OOV 1m Call, BS @ 200



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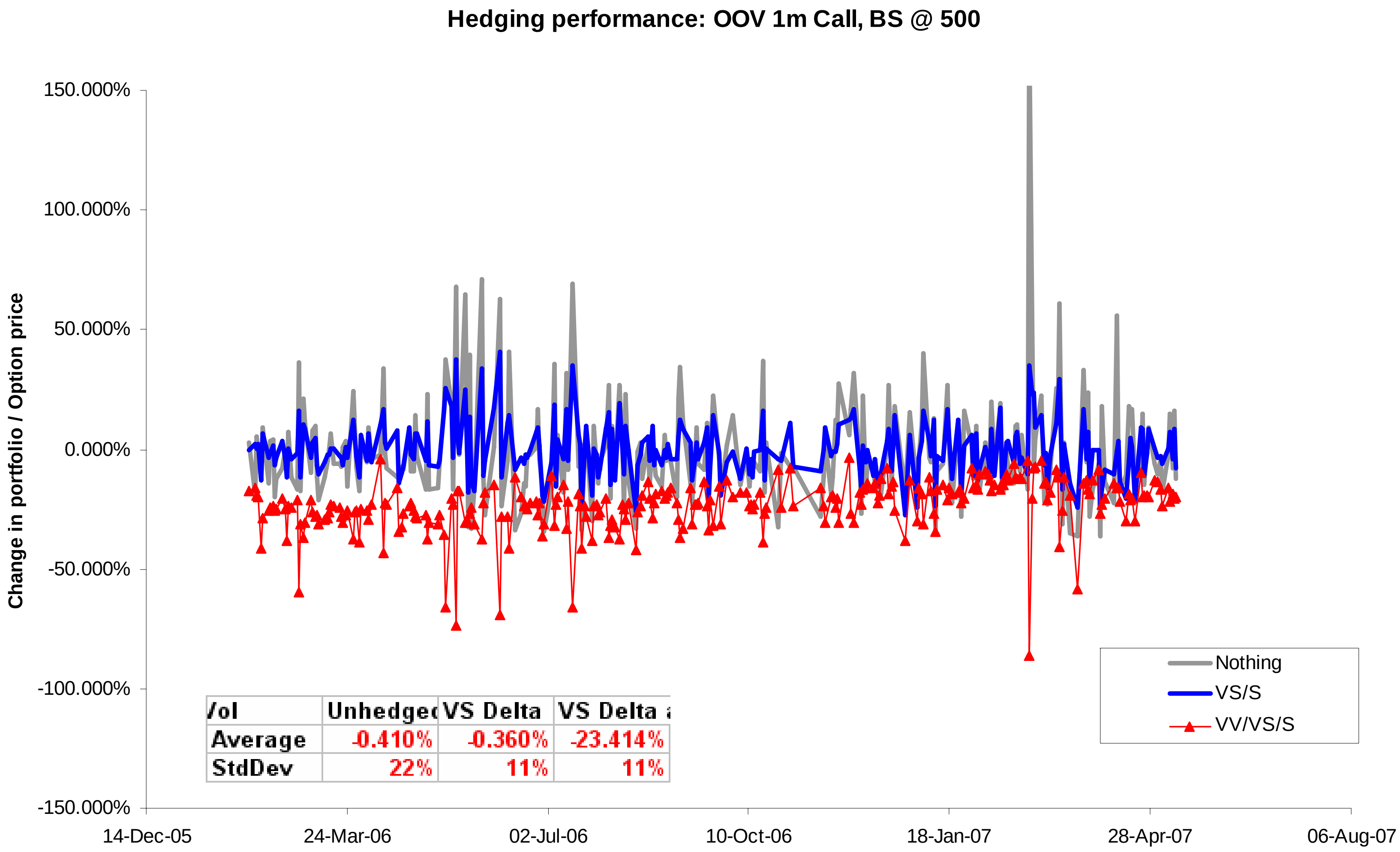
Options on Realized Variance

Measuring hedging performance



/ol	Unhedged	VS Delta	VS Delta a
Average	-0.410%	-0.360%	-23.414%
StdDev	22%	11%	11%

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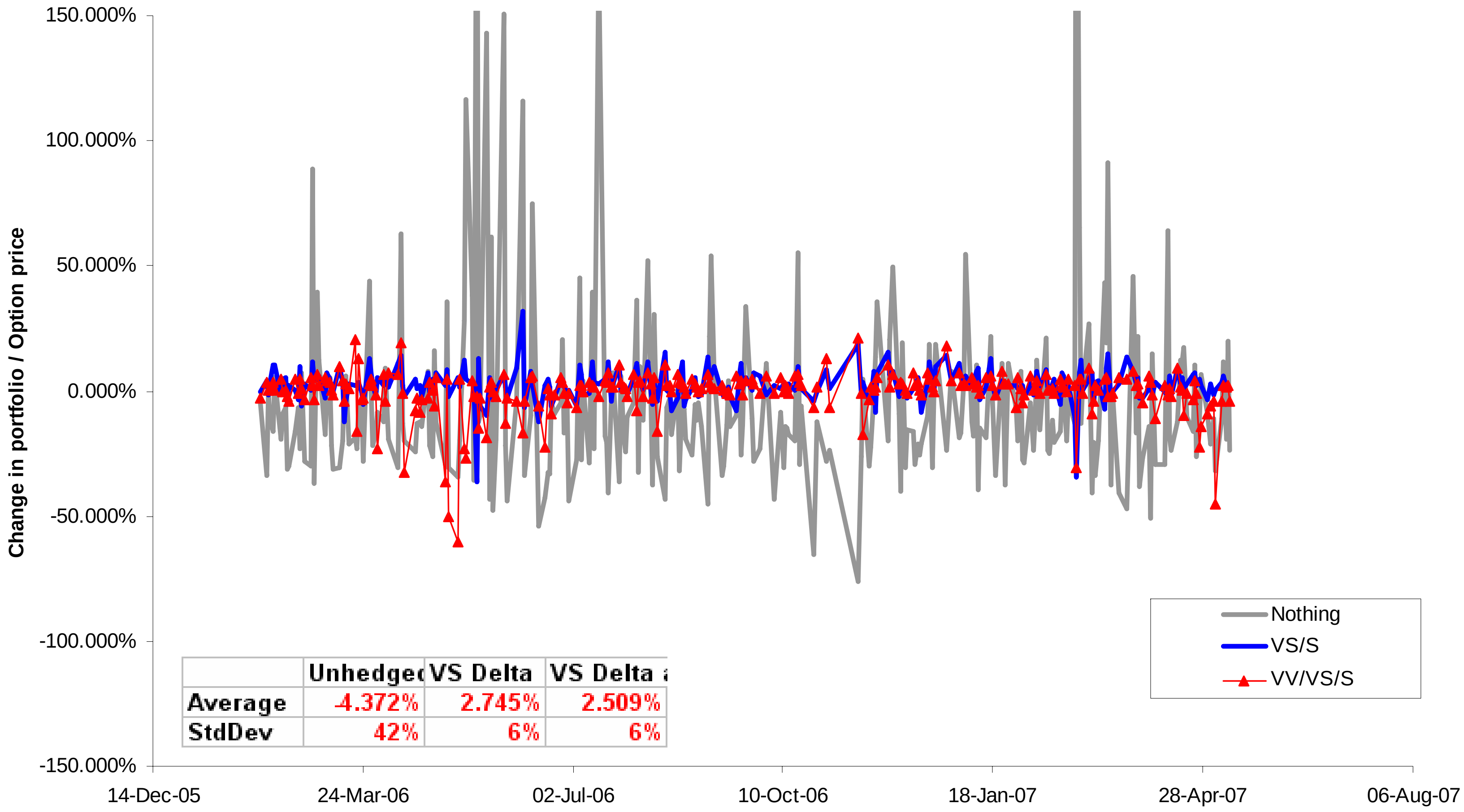
Options on Realized Variance

Measuring hedging performance

Hedging performance: OOV 1m Call, LN RevSpeed 5



	Unhedged	VS Delta	VS Delta a
Average	-4.372%	2.745%	2.509%
StdDev	42%	6%	6%

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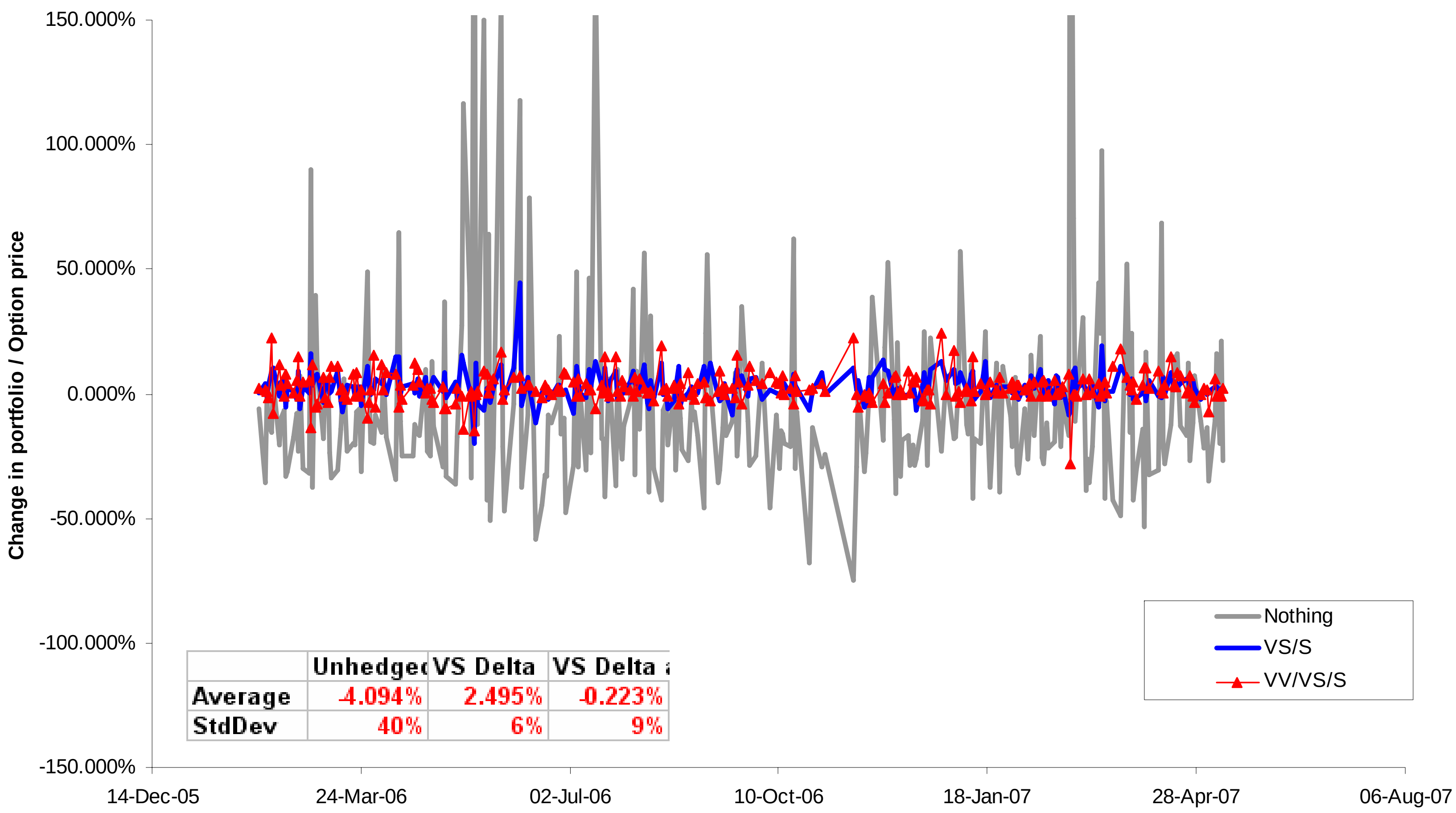
Options on Realized Variance

Measuring hedging performance

Hedging performance: OOV 1m Call, Heston RevSpeed 5



	Unhedge	VS Delta	VS Delta
Average	-4.094%	2.495%	-0.223%
StdDev	40%	6%	9%

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Options on Realized Variance

In Action - summary

- 1m ATM realized variance call

	Unhedged	VS Delta	VS & VolOfVol
BS	0.260%	-1.375%	-0.766%
LogNormal	-4.094%	2.495%	-0.223%
Heston	-4.372%	2.745%	2.509%

(Average)

	Unhedged	VS Delta	VS & VolOfVol
BS	28%	16%	22%
LogNormal	40%	6%	9%
Heston	42%	6%	6%

(StdDev)

- Results comparable to BS Asian on the equity (43%/13%/7%)

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Options on Realized Variance

In Action

Now 3m ATM realized variance call

– LogNormal and Heston Reversion Speed 2

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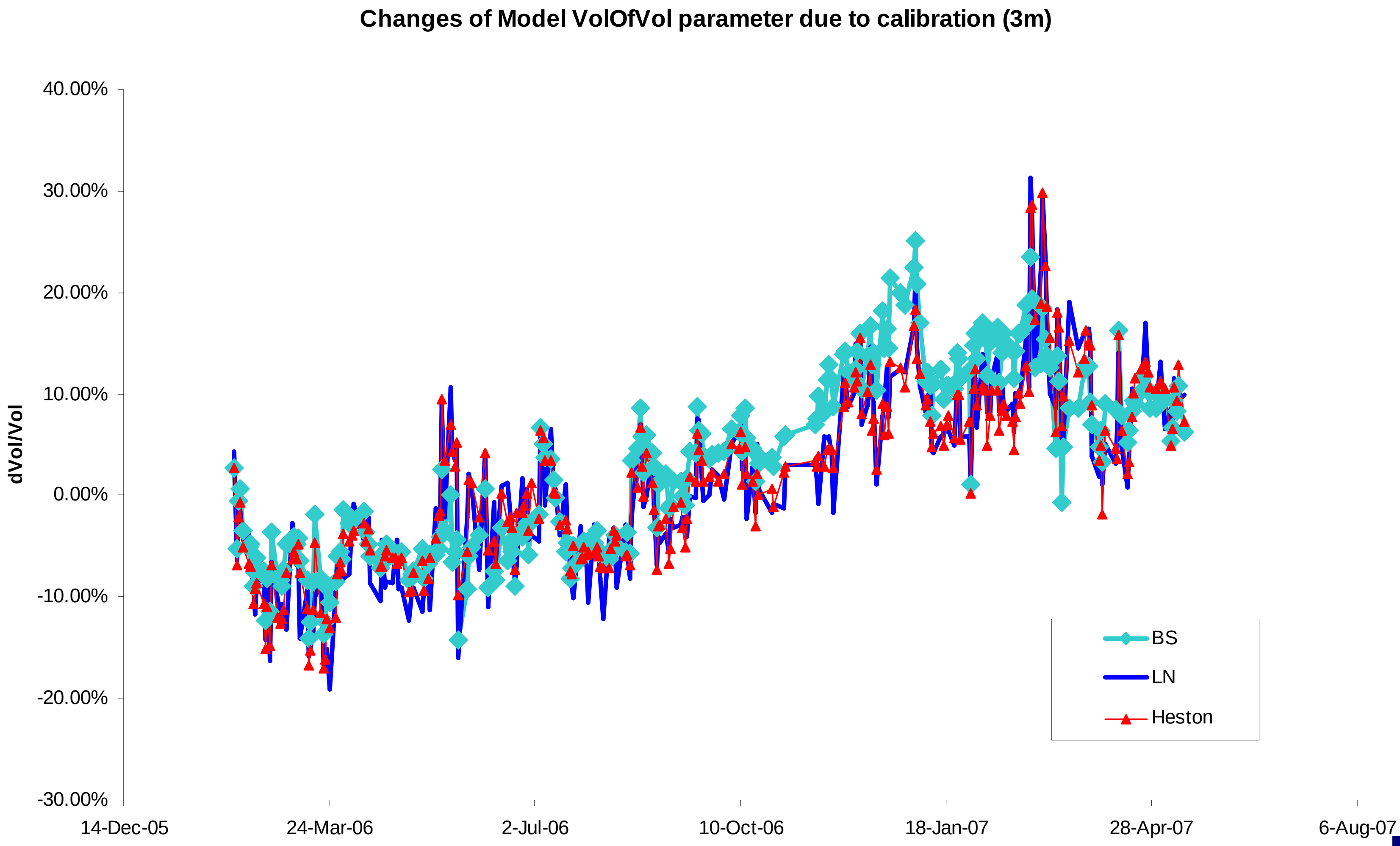
Options on Realized Variance

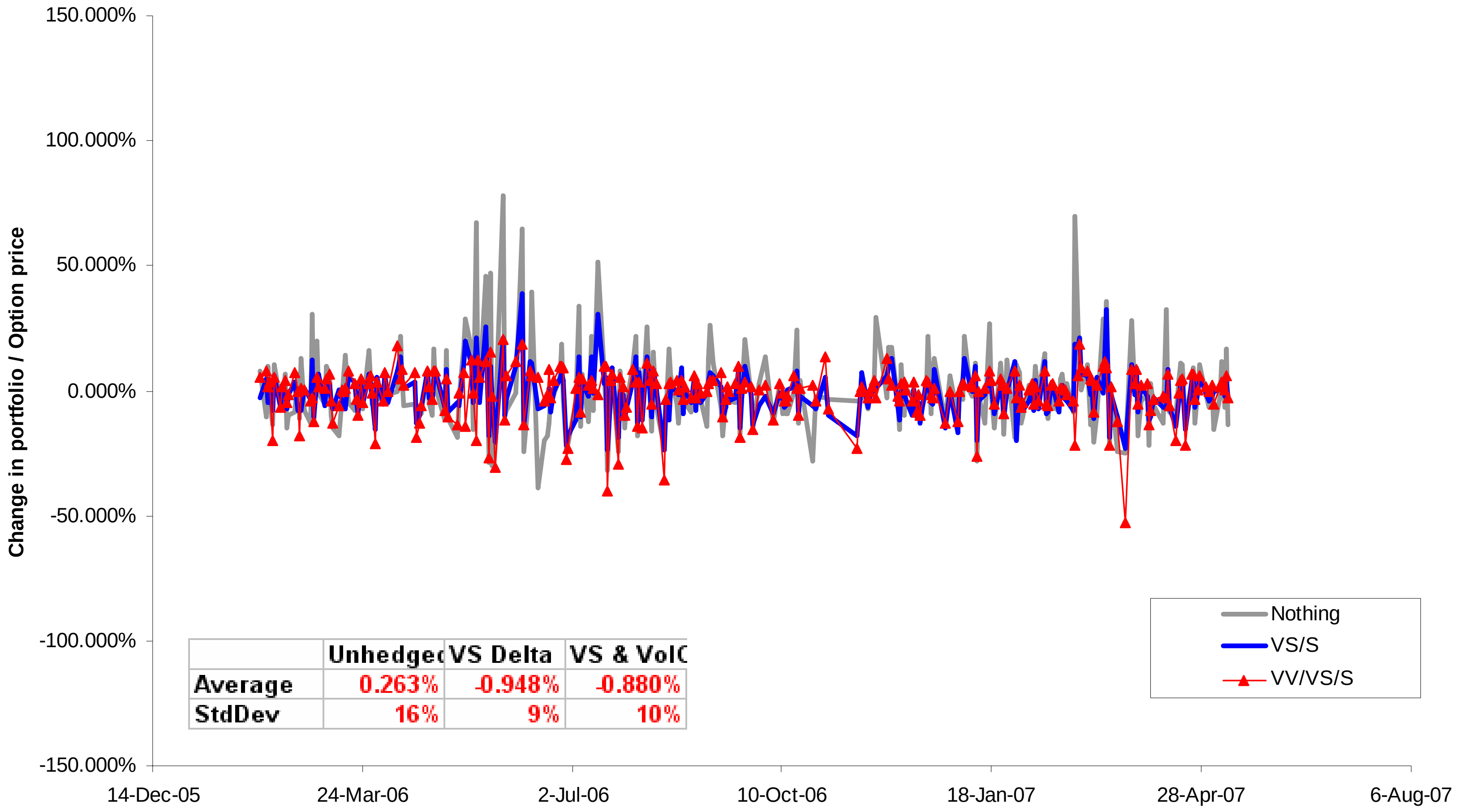
Measuring hedging performance



Options on Realized Variance

Measuring hedging performance

Hedging performance: OOV 3m Call, BS calibrated



	Unhedged	VS Delta	VS & VolC
Average	0.263%	-0.948%	-0.880%
StdDev	16%	9%	10%

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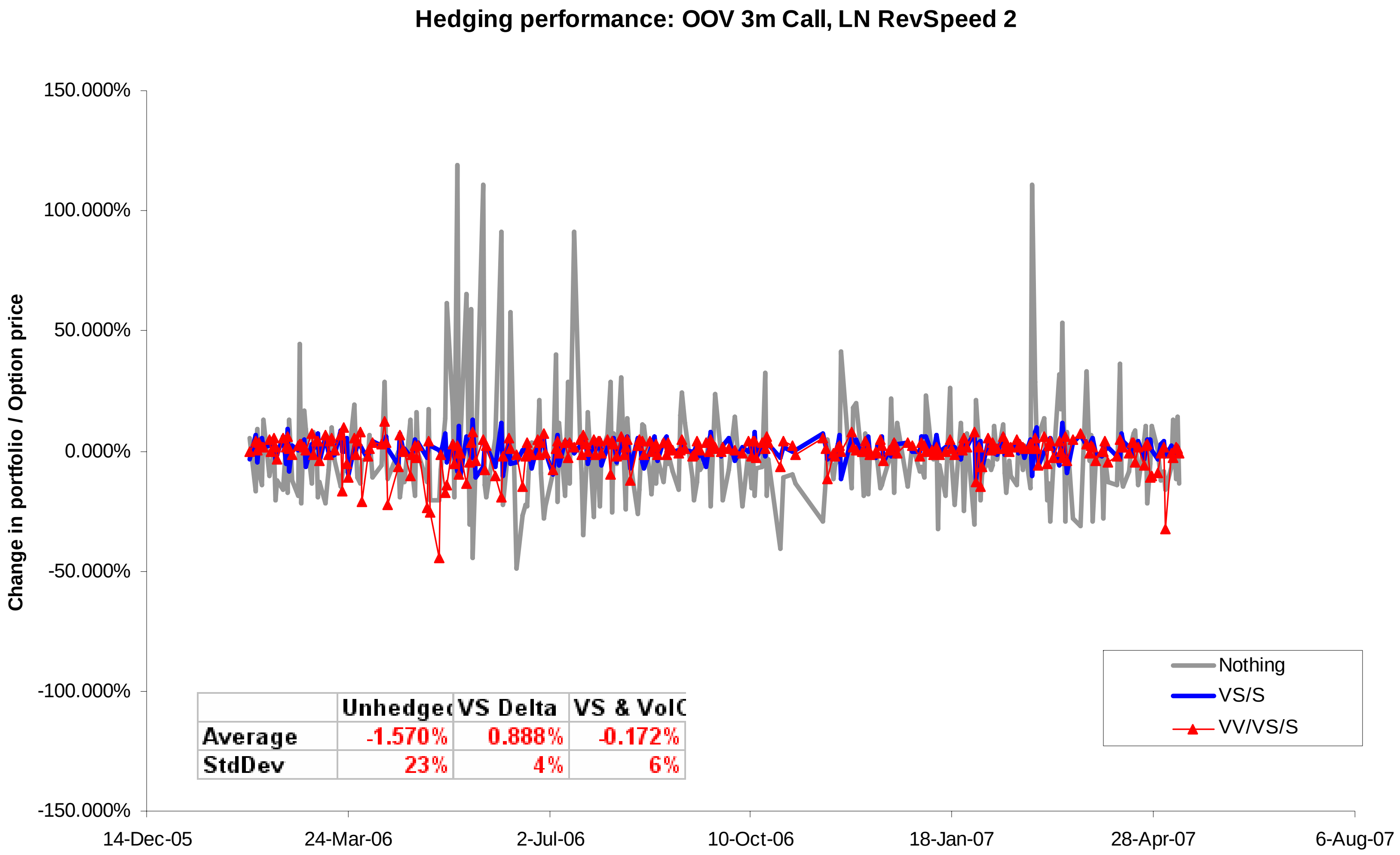
Options on Realized Variance

Measuring hedging performance



	Unhedged	VS Delta	VS & VolC
Average	-1.570%	0.888%	-0.172%
StdDev	23%	4%	6%

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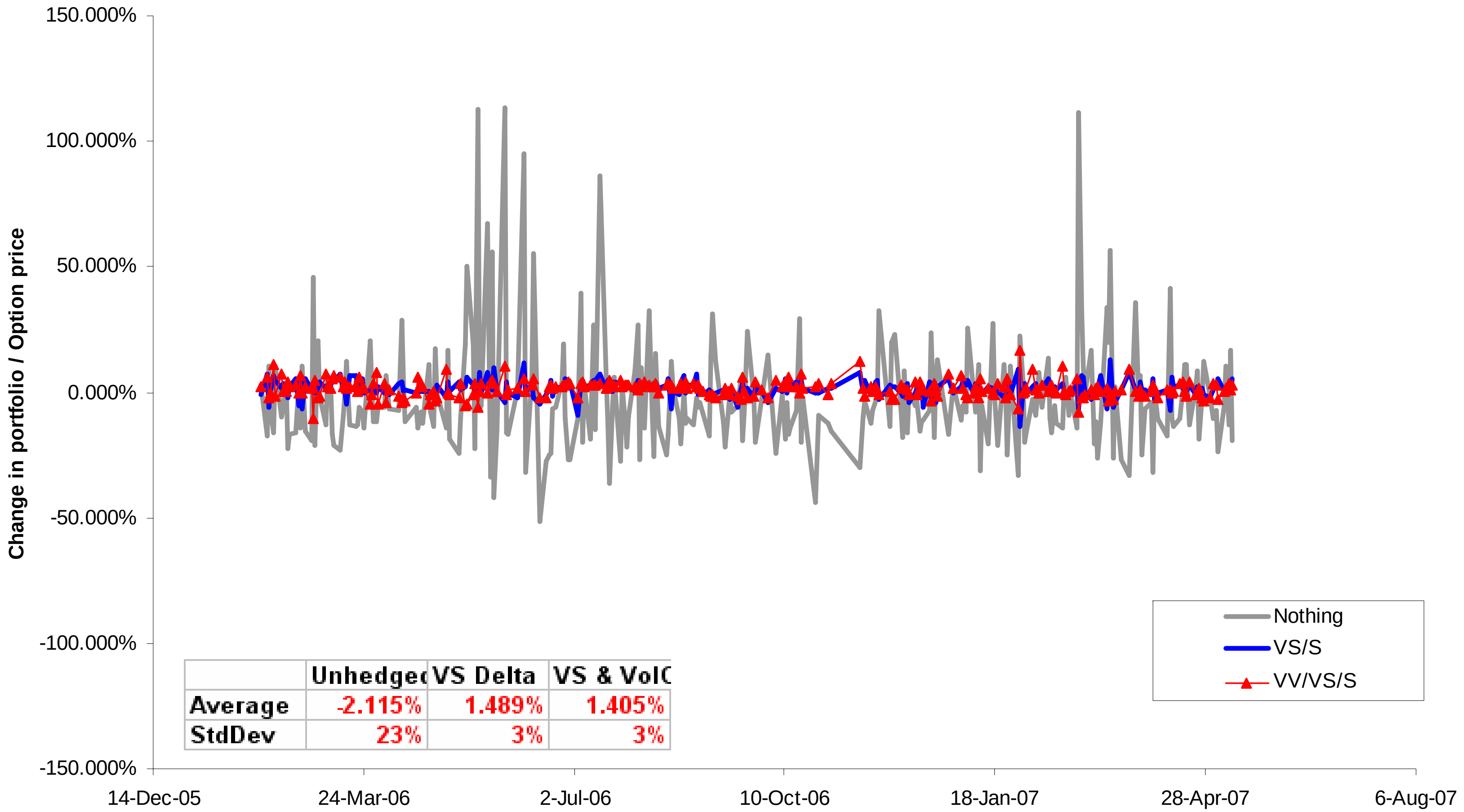
Options on Realized Variance

Measuring hedging performance

Hedging performance: OOV 3m Call, Heston RevSpeed 2



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Options on Realized Variance
In Action

- 3m ATM realized variance call

	Unhedged	VS Delta	VS & VolOfVol
BS	0.263%	-0.948%	-0.880%
LogNormal	-1.570%	0.888%	-0.172%
Heston	-2.115%	1.489%	1.405%

	Unhedged	VS Delta	VS & VolOfVol
BS	16%	9%	10%
LogNormal	23%	4%	6%
Heston	23%	3%	3%

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Options on Realized Variance
In Action

- Comments
  - The two fitted models do better than BS
  - Heston and LogNormal produce *very* similar results; cf. Buehler 2006
    $\rightarrow$ Heston is *much* faster, so we use this model.

- Questions:
  - Straddles ?
  - Impact of reversion speed
  - What happens ITM/OTM and what happens for longer maturities

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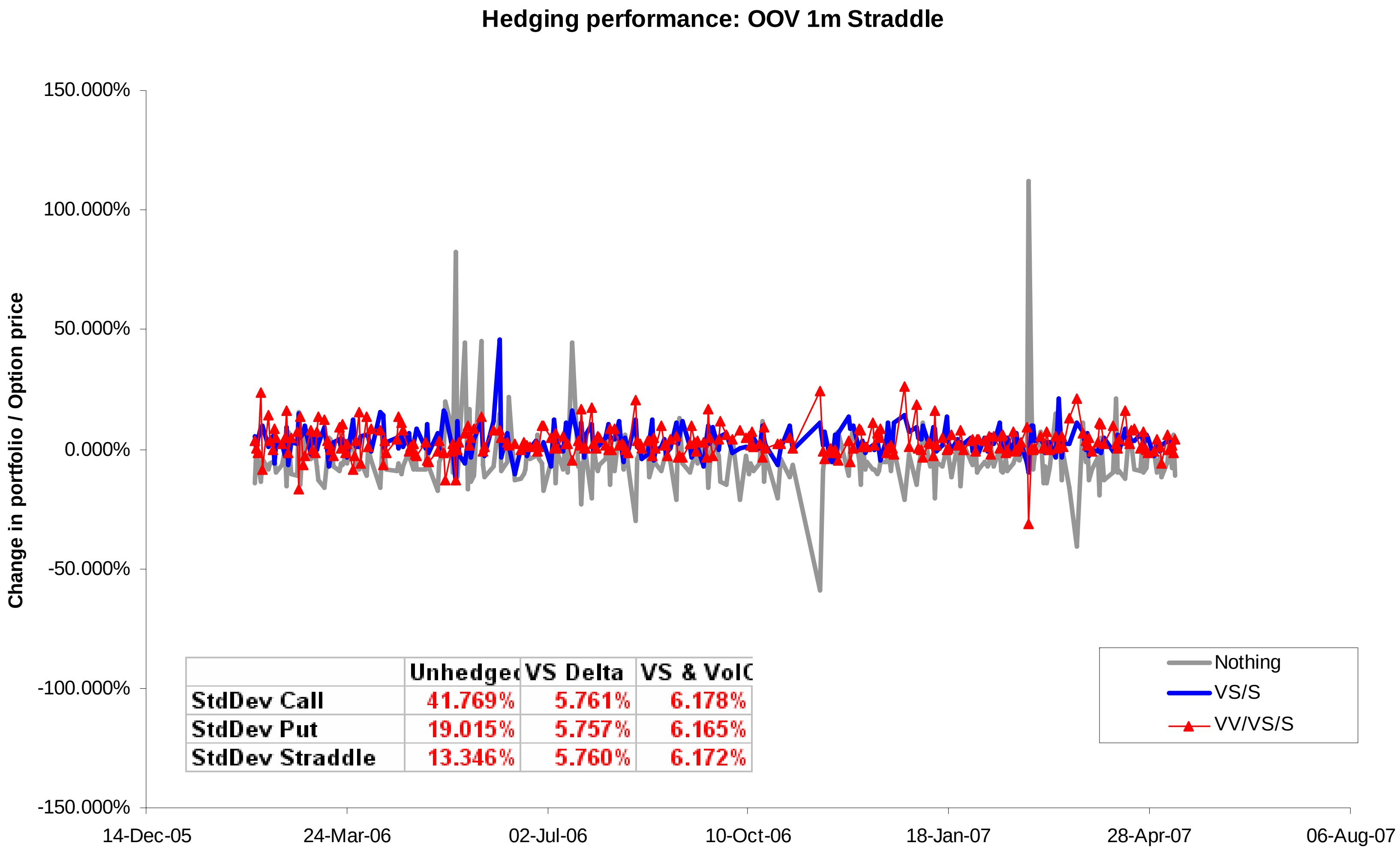
Options on Realized Variance

Questions – Straddles



	Unhedge	VS Delta	VS & Vol
StdDev Call	41.769%	5.761%	6.178%
StdDev Put	19.015%	5.757%	6.165%
StdDev Straddle	13.346%	5.760%	6.172%

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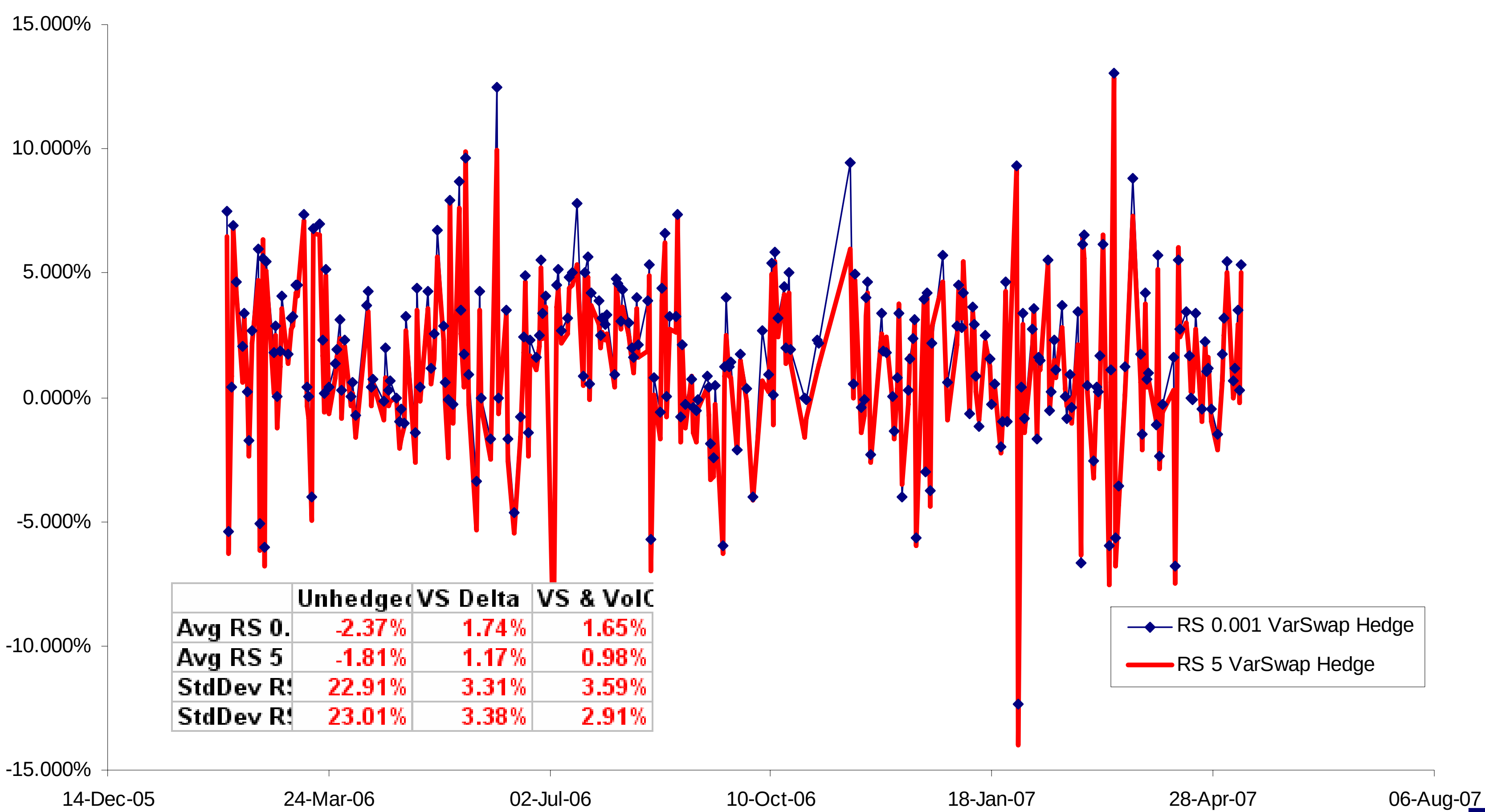
Options on Realized Variance

Questions– Impact of reversion speed

Hedging 3m ATM calls with Heston, RevSpeeds 0.001 and 5



	Unhedge(	VS Delta	VS & VolC
Avg RS 0.	-2.37%	1.74%	1.65%
Avg RS 5	-1.81%	1.17%	0.98%
StdDev R!	22.91%	3.31%	3.59%
StdDev R!	23.01%	3.38%	2.91%

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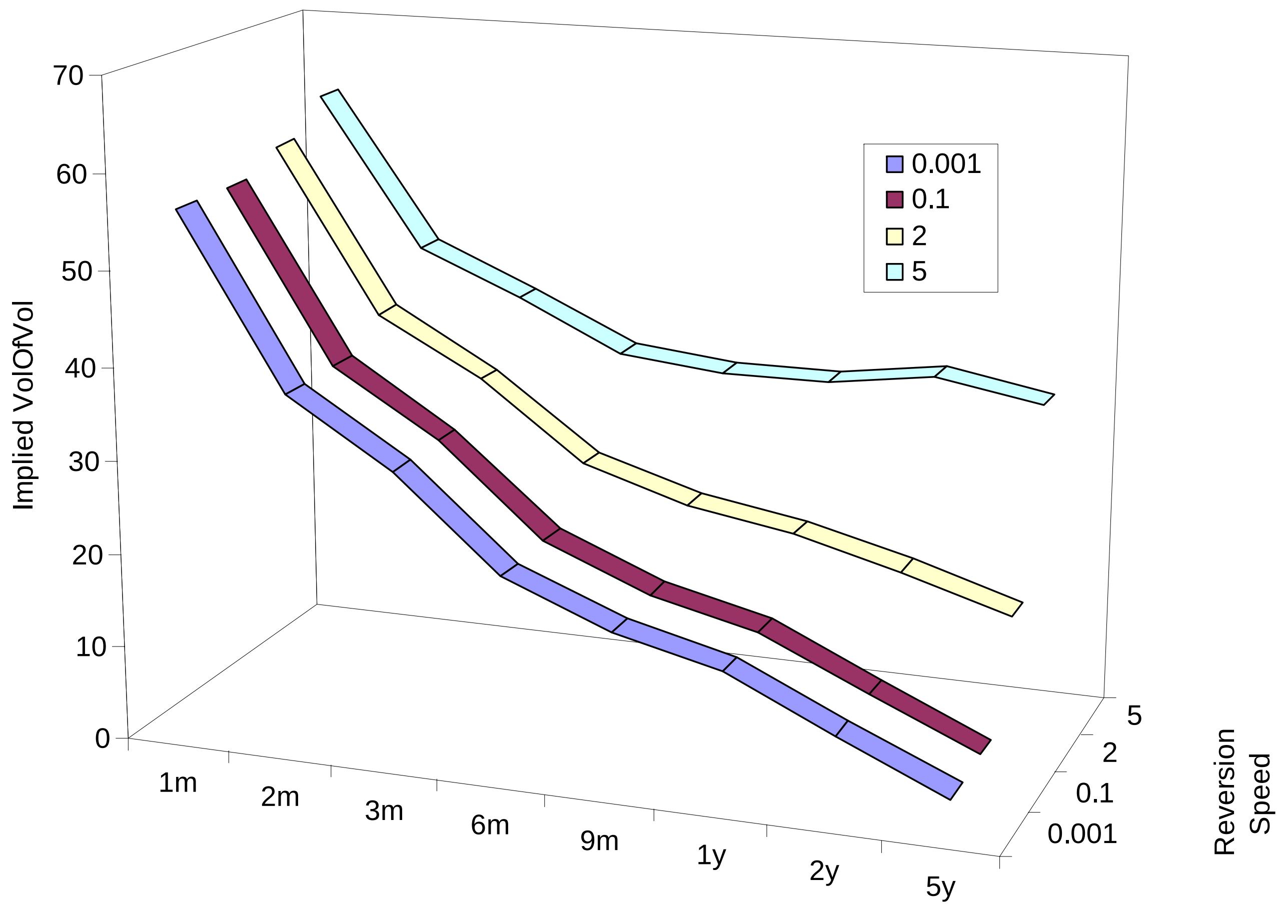
Options on Realized Variance

Questions– Impact of reversion speed

Average historic implied Heston VolOfVol for various reversion speeds



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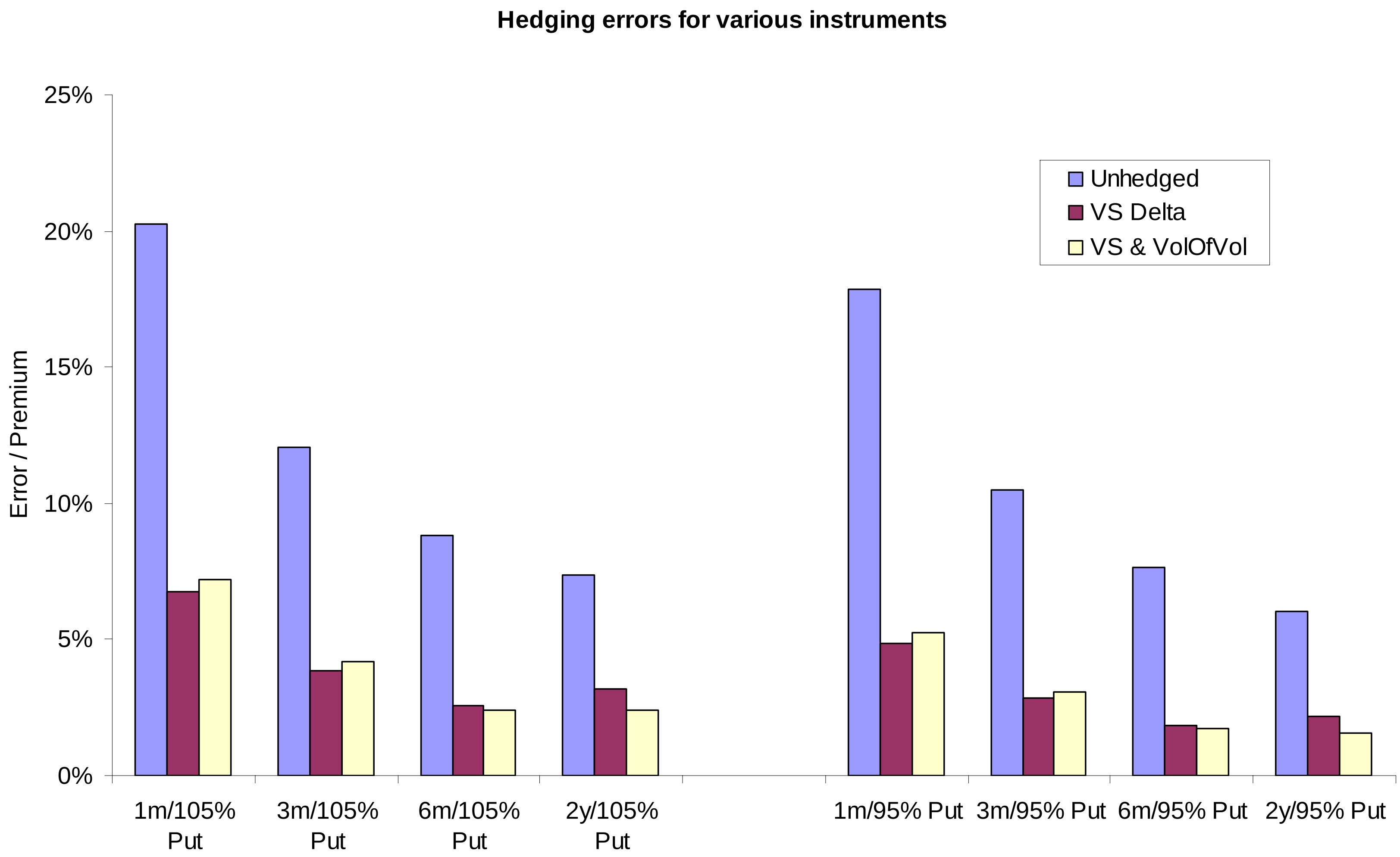
Options on Realized Variance

In Action – ITM/OTM and longer maturities



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Options on Realized Variance

In Action

- Summary
  - We have addressed the question “best model” by investigating the reduction of variance achieved by executing a VarSwapDelta-hedge and an additional VolOfVol-hedge.
  - Three models have been tested
    - BS-type model — bad performance for short maturities
    - Fitted Heston — reduction of around 80% of variance for calls realistic
    - Fitted Log-Normal — very similar to Heston
  - Fitted Heston
    - Reversion Speed unimportant when a single maturity is concerned
    - Semi-analytic European option prices

Thank you very much for your attention.
hans.buehler@db.com

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