Consistent Variance Curve Models

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Volatility as a traded asset

Introduction

- We are interested generally on pricing options on *realised variance* over business days $T_1 = t_0 < \cdots < t_n = T_2$,

$$RV(T_1, T_2) := \sum_{i=1}^{n} \left(\log \frac{S_{t_t}}{S_{t_{t-1}}} \right)^2$$

- We simplify and assume that realized variance is in effect the quadratic variation of the log of the stock price $S = (S_t)_t$ to the holder.
  - Quadratic variation is an unbiased estimator of the actual „realized variance", c.f. Barndorff-Nielsen et al (2004), i.e.

$$\left\langle \log S_{\bullet} / S_{T_1} \right\rangle \approx RV(T_1, T_2)$$

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Volatility as a traded asset

Introduction

- The basic product on realized variance is a *Variance Swap*.
  - The (zero strike) variance swap for the period $[T_1,T_2]$ pays the realized variance in this period to the holder.
  - We assume that variance swaps are liquidly traded. Their price process is denoted as

$$V(T_1,T_2) \equiv \left(V_t(T_1,T_2)\right)_{0\le t\le T_2}$$

Observation time t

$$V(T) := V(0,T)$$

  - At any time t also define the „floating maturity“ variance swaps for the period $[t,t+x]$ („Musiela-Parametrization” for variance swaps)

$$U_t(x) := V_t(t,t+x)$$

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Volatility as a traded asset

Exotic options on variance

- Since variance swaps are liquid, we do not need to model them.
  - They become input into our model.
- Products we are interested in
  - Vanilla options on variance

$$\left(RV(0,T) - kV_0(T)\right)^+$$

  - Forward-started options on variance

Price of the variance swap for [T,T+Δ] at time T

$$\left(RV(T,T+\Delta T) - kV_T(T,T+\Delta T)\right)^+$$

  - Options on variance swaps (c.f. also options on VIX)

Price of the variance swap for [T,T+Δ] today

$$\left(V_T(T,T+\Delta T) - kV_0(T,T+\Delta T)\right)^+$$

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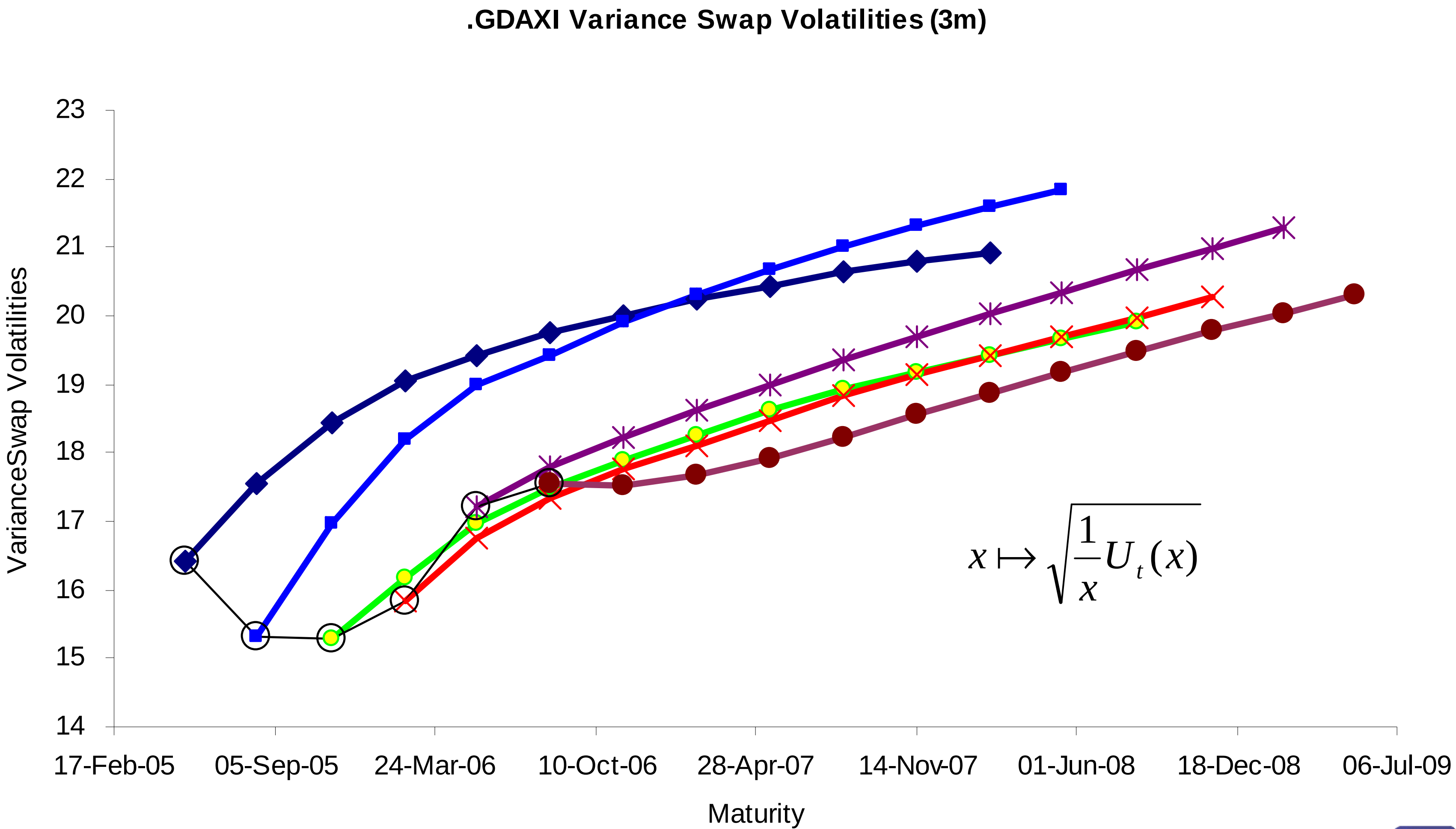
Variance Curve Models

… a picture of the market



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Options on Variance

Task list

- The idea is to develop a pricing model that describes the entire variance swap term structure.
  - Interest-rate type problem.
- Minimum requirements
  - Free of arbitrage.
  - Allows hedging of exotic options with variance swaps (ie, complete market).
  - Allows to model a corresponding stock price process.
  - Can be implemented.
- Note:
  We do not take into account European options on the underlying asset at this stage.

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Variance Swap Curve Models

Variance Curve Models
Modelling

- Assume (Ω,P,F) is a filtered probability space with d-dimensional extremal Browninan motion W.
- The idea is, to model the curve U as stochastic variable.
  The first problem is, that we need $x \rightarrow U_t(x)$ to be monotonic intreasing for all times t. Hence, we concentrate rather on *forward variance*

$$u_t(x) := \partial_x U_t(x)$$

  - Similar to forward rates in interest rate theory.
  - Note that forward variance cannot be negative, but might well be zero.

Variance Curve Models

Modelling

- <u>Idea 1: General HJM approach</u>
  Let $u=(u(x))_{x\geq 0}$ be a family of processes $u(x)=(u_t(x))_{t\geq 0}$ which are given as

$$du_t(x) := a_t(x)dt + \sum_{j=1,\ldots,d} b_t^j(x)dW_t^j$$

  (a and b are some suitably integrable and previsible processes).

- <u>Problem 1:</u>
  When is u a „*variance curve model*“ in the sense that we can define a local martingale S, such that the market with S and the variance swaps is free of arbitrage?

- Answer (cf later): essentially if $v(T)$ with $v_t(T):=u_t(T\text{-}t)$ is a local martingale, i.e.

$$a_t(x) = \partial_x u_t(x)$$

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Consistent Variance Curve Models

Next step

- However, HJM-type processes are by far too general:
  - It is much more natural to model u via a finite-dimensional Markov-process (in particular if we want to actually implement such a model).
  - Let us assume we write u as a function G of an m-dimensional Diffusion $Z=(Z_t)_{t\geq 0}$,

$$u_t(x) := G(Z_t; x).$$

- <u>Problem 2:</u>
  Given a "nice" function G, which diffusion Z is "consistent" with G in the sense that $G(Z; \cdot)$ is a variance curve model ?
  - Consistency problem a'la Bjoerk et al for interest rates.

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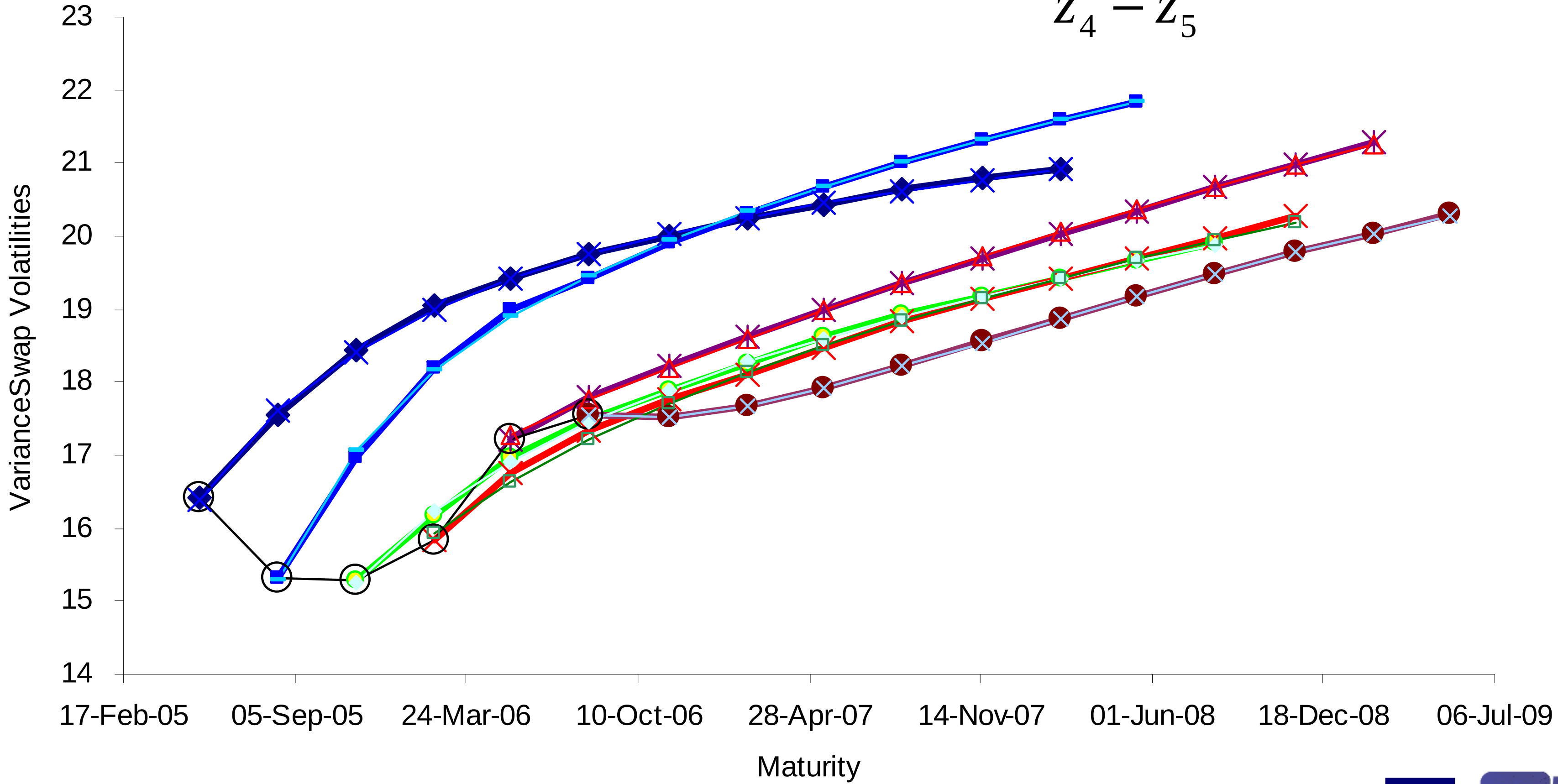
Fitting a function to the market

$$G(z;x) = z_3 + (z_1 - z_3)e^{-z_4 x} + (z_2 - z_3)\frac{z_4}{z_4 - z_5}\left(e^{-z_5 x} - e^{-z_4 x}\right)$$



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Consistent Variance Curve Models

Definition

- Definition

  We call a non-negative $C^{2,2}$-Function $G\!:D\times\mathrm{R}_{\geq 0}\rightarrow\mathrm{R}_{\geq 0}$ a *variance curve functional* if

$$\int_0^T G(z;x)dx < \infty$$

  for all T and $z\in D$ where D is an open set in $R_{\geq 0}{}^m$.

- We denote by Ξ the set of coefficients $C=(\mu,\sigma)$ for which the SDE

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\ldots,d} \sigma^j(Z_t)dW_t^j$$

  has a unique, non-explosive m-dimensional strong solution for each starting point $Z_0\in D$. We call such a process *parameter process.*

  – Note that time-dependent coefficients are part of this definition.

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Consistent Variance Curve Models

Definition

- <u>Definition</u>
  We call $C=(\mu,\sigma)\in\Xi$ *consistent* with G if

$$u_t(x) := G(Z_t; x)$$

  defines for all $Z_0\in D$ a variance curve model. Then, (C,G) is called a *consistent variance curve model*.

- <u>Proposition</u>
  This is the case, iff Z remains in D and if the “heat equation”

$$\partial_x G(z;x) = \mu(z)\partial_z G(z;x) + \frac{1}{2}\sigma^T\sigma(z)\,\partial^2_{zz}G(z;x)$$

  is satisfied (note that this is a condition on $C=(\mu,\sigma)$ and not on G).

Consistent Variance Curve Models

Associated stock price process

- The associated price process and the Markov-property
  Given a consistent variance curve model $C=(\mu,\sigma)\in\Xi$ and a "correlation function" $\rho:\mathrm{R}_{\geq 0}\times D\rightarrow[-1,1]^d$ with $|\rho|_2=1$, define

$$dS_t = \sum_{j=1,\dots,d} S_t \rho^j(S_t;Z_t)\left\{\sqrt{G(Z_t;0)}dW_t^j\right\}.$$

- The solution S for this SDE always exists, is a local martingale and the system (S,Z) is Markov.
  - The Markov-property allows discussion of completenes.
  - The correlation function controls the skew of the implied vol.
  - Note that the freedom of chosing ρ might be limited if S is to be a true martingale.
- The "short variance" of the stock price is

$$G(Z_t;0)$$

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Consistent Variance Curve Models

Absence of arbitrage

- <u>Theorem</u>
  For each variance curve model u and any correlation function ρ, the market (S,V) of the associated stock price and the variance swaps $V=V(T)_{T\geq 0}$

$$V_t(T) := G(Z_t; T-t) + \int_0^t G(Z_s; 0), ds.$$

$U_t(T\text{-}t)$

Running variance

  is free of arbitrage.

  – The simplicity of this condition is a strong advantage to related approaches such as „Stochastic Implied Volatility“.

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Hedging

General Markovian markets

- Setting
  Assume we have a market with K tradable instruments $X=(X^1,\ldots,X^K)$ and an additional finite variation process $A=(A^1,\ldots,A^L)$, such that $Y=(X,A)$ is the unique, strong non-explosive solution of

$$\begin{cases} dX_t &= \sum_{j=1,..,d} \Omega^j(X_t, A_t)dW^j \\ dA_t &= C(X_t, A_t)dt \end{cases}$$

  – The process A can encode time or, for us, running variance.

- Problem 3
  When is the market of all "hedging-relevant" payoffs on Y complete, i.e. the market of all F^Y-measurable non-negative L^1 random variables ?

Hedging

General Markovian markets

- We say that Y "weakly preserves smoothness", if

$$\mathrm{P}F(t;x,a) := \mathrm{E}\left[F(X_t, A_t) \mid X_0 = x, A_0 = a\right]$$

  maps smooth F with compact support into C^1-functions in x.

- Theorem ("Delta-hedging works"):
  If the above holds, then the market of all "hedging-relevant" payoffs is complete.

- Proposition:
  In the case that Ω und C are differentiable with locally Lipschitz derivatives, this holds.
  In particular, the volatility matrix may not have full rank.
  – Open question: can we relax this assumption further?

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Hedging

Consistent variance curve models

- This can be used for consistent variance curve models

$$dZ_t = \mu(Z_t)dt + \sum_{j=1,\dots,d} \sigma^j(Z_t)dW_t^j$$

$$dA_t := G(Z_t;0)dt$$

- We need an additional invertibiliy criterion on G so that we can recover the state Z locally from the market.

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Hedging
Consistent variance curve models

- Then, an option

$$H_t := \mathrm{E}\left[H\left(\int_0^T \zeta_s ds \right) | \mathrm{F}_t \right]$$

can be replicated with variance swaps, i.e. “VarSwapDeltas” $\nu^1,\ldots,\nu^m$ exist such that

$$dH_t = \sum_{k=1}^{m'} \nu_t^k dV_t(T_k)$$

- If H_t is given as a differentiable function $H_t=h(t;V_t,A_t)$, then we have

$$\nu_t^k = \partial_{V^k} h(t;V_t,A_t)$$

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Applications

Consistent Variance Curve Models

Double Heston

- The example we had before

$$G(z;x) = z_3 + (z_1 - z_2)e^{-\kappa x} + (z_2 - z_3)\frac{\kappa}{\kappa - c}\left(e^{-cx} - e^{-\kappa x}\right)$$

  – The parameters k and c must be constant to avoid arbitrage.

  – A consistent factor model for this G must have the form

$$\begin{aligned} dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \sigma_1(Z_t)dW_t \\ dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \sigma_2(Z_t)dW_t \\ dZ_t^3 &= \sigma_3(Z_t)dW_t \end{aligned}$$

Note that non-negativity must be ensured.

Consistent Variance Curve Models

Double Heston

- This works

$$
\begin{aligned}
dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \nu {Z_t^1}^{\alpha} d\hat{W}_t^1 \\
dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \mu {Z_t^2}^{\beta} d\hat{W}_t^2 \\
dZ_t^3 &= \eta Z_t^3 d\hat{W}_t^3
\end{aligned}
$$

for a correlated vector of Brownian motions and $\alpha,\beta > 1/2$.*

  – Complete model where S and the variance swaps are true martingales.

- Calibration strategy
  – First fit the functional G, i.e. find Z_0.
  – The use the European prices to infer the correlation/volatility structure using Monte-Carlo calibration.
  – Pretty expensive.

* In fact, instead of x^α we use $(x+\varepsilon)^\alpha - \varepsilon^\alpha$.

Consistent Variance Curve Models

Double Heston

- Alternative

$$
\begin{aligned}
dZ_t^1 &= \kappa(Z_t^2 - Z_t^1)dt + \nu\sqrt{Z_t^1}\,d\hat{W}_t^1 \\
dZ_t^2 &= c(Z_t^3 - Z_t^2)dt + \mu\sqrt{Z_t^2}\,dW_t^2 \\
Z_t^3 &= const
\end{aligned}
$$

with W^2 independent of W^1 and stock price Browninan motion.

– Affine model.

– Fourier-pricing of Europeans via Ricatti equations.

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Consistent Variance Curve Models

Fitting the market

- A good functional may fit well, but probably never perfect.
  - We need “post-fitting”.
- Assume that $(C=(\mu,\sigma), G)$ is a consistent pair and assume that we observe in the market a short variance curve $m_0(x)$.
  Then,

$$u_t(x) := \frac{m_0(t+x)}{G(Z_0; t+x)} G(Z_t; x) \equiv \frac{m_0(t+x)}{\mathrm{E}\left[\tilde{u}_t(x)\right]} \tilde{u}_t(x)$$

“fits the market” in the sense that

$$\mathrm{E}\left[v_T(T)\right] = \mathrm{E}\left[u_T(0)\right] = m_0(T)$$

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Consistent Variance Curve Models

Fitting the market

- For the affine example before, the result is still affine and can be treated with Ricatti.
- Another example (Dupire)

$$G(z;x) = \exp\left\{ ze^{-kx} + \frac{s^2}{4}\left(\frac{1-e^{-2kx}}{2k}\right)\right\}$$

  – Log-normal short variance fitted to the variance swap market.

  – Two-factor version cf. Bergomi.

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Consistent Variance Curve Models

Conclusions

- The framework
  - Provides means to ensure absence of arbitrage and market completeness.
  - Concept of “consistency” is a desirable property from a conceptual point of view (in Buehler 2006, we also discuss applications of variance curves in Hilbert spaces a’la Filipovic/Teichmann).
  - From a practical point of view, “post-fitting” to the market ensures that we fit the variance swaps perfectly.
- To keep in mind
  - Real markets price in jump risk in the underlying (there is an upward implied vol slope in options on realised variance).
  - When pricing options, use “proper” realized variance, not quadratic variation!

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Thank you very much for your attention.
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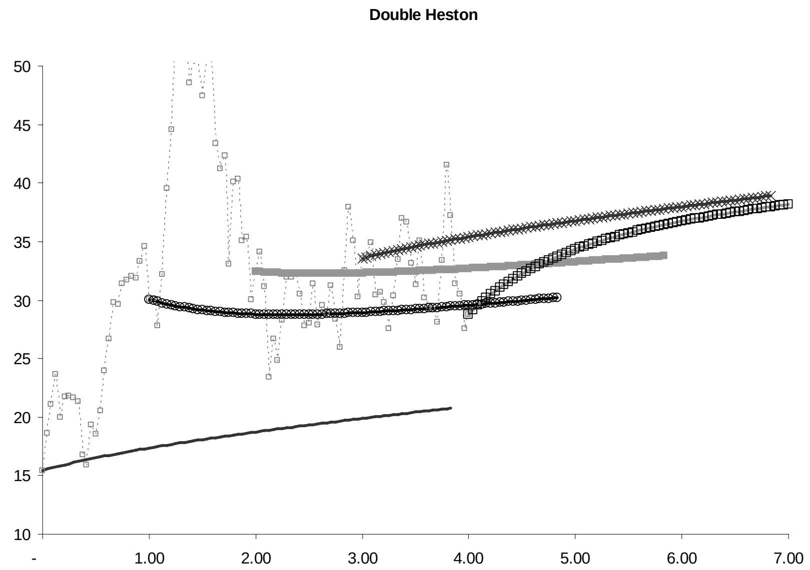
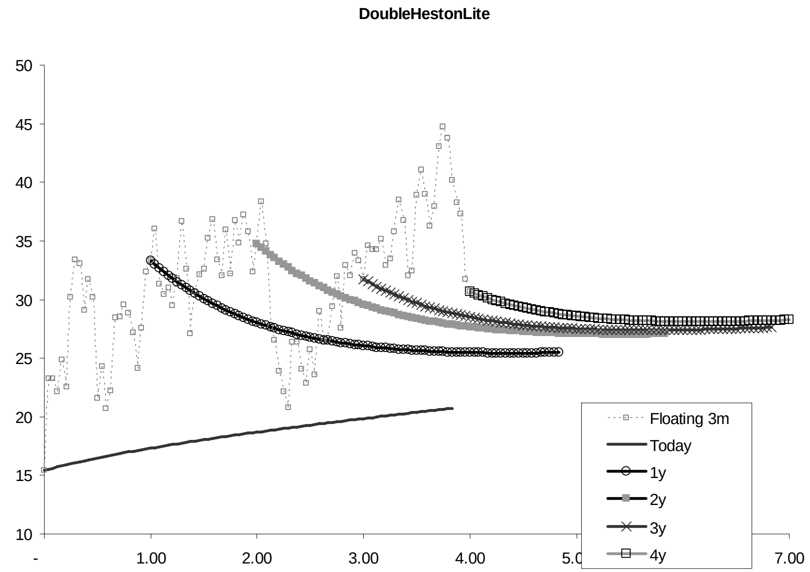
Consistent Variance Curve Models

Double Heston – effect of stochastic "mean reversion level"





The long term level of the variance swap curve can move in the three-factor model.
The right hand side shows the one-factor version.

(May 2 data)

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