Levy Models in Option Pricing

Utilising Volatility Smile Models to Optimise Pricing and Hedging Strategies

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Contents

- Some examples

Example: Brownian motion with drift.

- A Gaussian Process X is a continuous process with deterministic covariance matrix.
  - As a consequence, it has independent increments.
  - If it is centred and has *stationary* increments, it can be written as

$$X_t = \gamma t + \sigma W_t$$

    for a vector γ, the root σ of the covariance matrix $A = \sigma^T \sigma$ and a standard Brownian motion W.
  - For each fixed T and each n, we can write X as a sum of independent processes of the same form,

$$X_T = X_{1/T} + \cdots + X_{n/T}$$

Example: Poisson Process (1)

- A Poisson-process is a process which jumps by 1 with exponentially distributed times between two jumps.
  - Let $(T_i)_i$ be a sequence of stopping times such that $\tau_i := T_i - T_{i-1}$ is an iid sequence of exponentially distributed random variables with intensity λ: $\mathrm{P}[\tau > t] = e^{-\lambda t}$
  - Then,

$$M(\omega,(a,b]) := \#\{n : a < T_n(\omega) \le b\}$$

is called a *random jump measure* which counts the number of stopping times between a and b. The process

$$N_t(\omega) := M(\omega,[0,t])$$

is then a *Poisson process with intensity* λ . It counts the jumps until t.

  - By construction,

$$N_t(\omega) := M(\omega,[0,t]) \equiv \int_{[0,t]} M(\omega,dx) = \sum_{\mathrm{n}} 1_{\{T_n \le t\}} = \#\{n : T_n \le t\}$$

Example: Poisson Process (2)

- We have

$$P[N_t = n] = e^{-\lambda t}\frac{(\lambda t)^n}{n!}$$

- Moreover, N is a Markov process with stationary and independent increments.

Example: Compensated Poisson Process

- A *compensated* Poisson process is a Poisson-process adjusted to be martingale.
  - Since a Poisson process has independent increments, it is enough to ensure that the expectation of the increments is compensated.
  - Hence, if N is a Poisson-process, then

$$\tilde{N}_t := N_t - \mathrm{E}[N_t] = N_t - \lambda t$$

is a martingale with independent and stationary increments.

Example: Compound Poisson Process (1)

- A *compound Poisson process* is a Poisson-process with jumps of some distribution F.
  - Let N be a Poisson process with intensity λ and let $(Y_i)_i$ be an iid sequence with distribution F. Then,

    $$X_t := \sum_{i=1,\dots,N_t} Y_i$$

    is called a *compound Poisson process with intensity* λ *and distribution* F.
  - The process jumps N_t times with jumps given via F.
  - Its characteristic function is given by

    $$\mathrm{E}[e^{izX_t}] = \exp\{\lambda t(\hat{\mathrm{F}}(z) - 1)\}$$

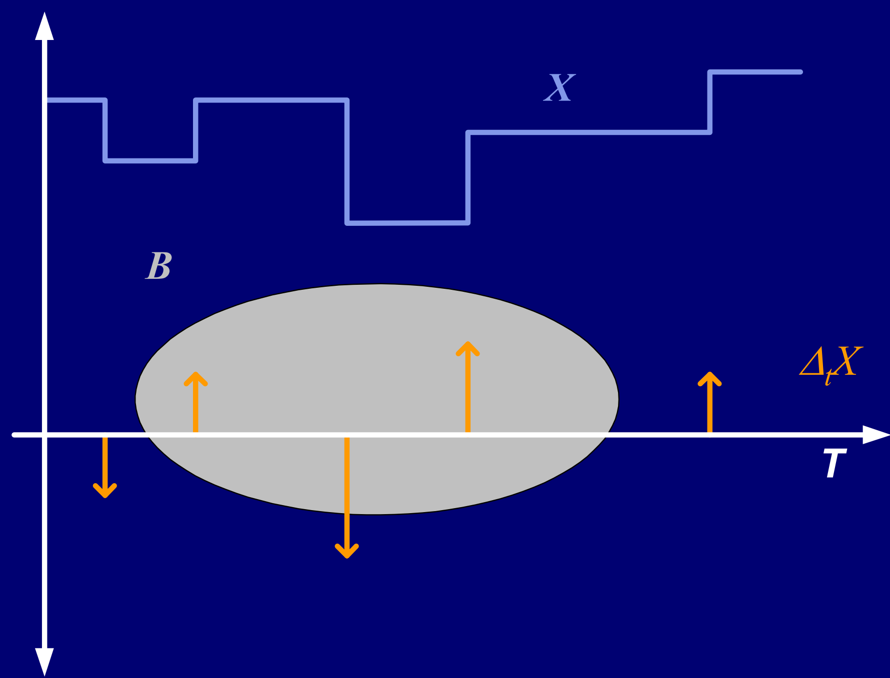
    where $\hat{\mathrm{F}}(z) := \mathrm{E}[\mathrm{e}^{izY_1}]$ denotes the characteristic function of F.

Example: Compound Poisson Process (2)

– The *jump measure* of a compound poisson process is the measure

$$J[\omega,(a,b]\times B^2] := \#\left\{(t,\Delta_t X(\omega)) \in (a,b]\times B^2 \setminus \{0\}\right\}$$

defined on the product space for measurable sets $B \subset [0,\infty)\times\Re^d$. It measures how often a *jump sequence* is within some set B.



Example: Compound Poisson Process (2)

– We then have

$$X_t(\omega) = \sum_{s \le t : \Delta_t X(\omega) \neq 0} \Delta_t X(\omega) = \int_{[0,t] \times \Re^d \setminus \{0\}} x \, \mathrm{J}[\omega, d(s,x)]$$

- The first equality is clear since the process has no continuous part.
- The second is just rewritten in terms of the above measure.
- For a Poisson-process,

$$\mathrm{J}[\omega, B^1 \times B^2] := M(\omega, B^1) 1_{1 \in B^2}$$

Example: Compound Poisson Process (4)

- As a generalisation of the standard case, we call μ with

$$\mathrm{E}[\mathrm{J}[A]] = \mu(A)$$

the *intensity measure* of J.

- Hence,

$$\tilde{\mathrm{J}}[\omega, B] := \mathrm{J}[\omega, B] - \mathrm{E}[\mathrm{J}[B]] = \mathrm{J}[\omega, B] - \mu[B]$$

is a random measure with vanishing Expectation (in ω), and the process

$$\tilde{X}_t(\omega) := \int_{[0,t]\times\Re^d\setminus\{0\}} x\,\tilde{\mathrm{J}}[\omega, d(s,x)] = X_t(\omega) - \int_{[0,t]\times\Re^d\setminus\{0\}} x\mu[d(s,x)]$$

is centred martingale (the compensated process). For the example before,

$$\tilde{X}_t = \sum_{i=1,\ldots,N_t} Y_i - \lambda t\,\mathrm{E}[Y_1]$$

Levy Processes - idea

- What are the common properties of the preceding examples?
  - Independent and
  - stationary increments.
  - We can represent each X_T as a sum of iid variables.
- Can we generalise this ?

- Levy processes

Definition: Levy Processes

- Definition
  - A cadlag process $X = (X_t)_{t\geq 0}$ is called a *Levy process* iff
    - it has independent and stationary increments and if
    - it is "stochastic continuous", ie the probability of a jump at some fixed time t is zero:

$$\lim_{h\downarrow 0} \mathrm{P}[|X_{t+h} - X_t| \geq \varepsilon] = 0$$

- Observations
    - A continuous process with independent increments is Gaussian. As mentioned in the beginning, stationary increments imply that this means that it is a Brownian motion with drift.
    - A linear combination of independent Levy processes is a Levy process.

The Levy measure

- <u>Definition</u>
  - The *Levy measure* ν of a Levy process X is given by

$$\nu[A] := \mathrm{E}[\#\{t \leq 1 : \Delta_t X \in A \setminus \{0\}\}]$$

ie it is the expected number of jumps before 1 with a size which belongs to A.

- The measure ν is a Radon measure with $\int (|x| \wedge 1)^2 \nu(dx) < \infty$
- The jump-measure J of X has intensity measure

$$\mu[(a,b] \times \tilde{A}] := \lambda \int_a^b \nu(\tilde{A})\, dt$$

that is

$$\mathrm{E}[\mathrm{J}[A]] = \mu(A)$$

If X has no Diffusion and no Drift part, then the compensated process

$$\tilde{X}_t := \sum_{s \leq t; \Delta_s X \neq 0} \Delta_s \tilde{X} = \int_{[0,t] \times \Re^d \setminus \{0\}}^t x\, \tilde{\mathrm{J}}[d(s,x)] = X_t - t \int_{\Re^d \setminus \{0\}} x \nu(dx)$$

is a martingale.

Levy-Ito decomposition and the characteristic triple (1)

- Idea
  - We know already that a continuous Levy process is a Gaussian process. What is then the impact of jumps? Can we decompose a Levy process with arbitrary paths?
    - When can we write X as continuous part plus jumps:

$$X_t = \gamma t + \sigma W_t + \sum_{s \le t} \Delta_s X = \gamma t + \sigma W_t + \int_{[0,t] \times \Re^d} x \, \mathrm{J}[d(s,x)]$$

    - A problem arises if the number of jumps is not finite per interval $[0,t]$ - we have to ensure existence of the integral. The problem are “many” small jumps (many big jumps are impossible, if X is cadlag).
    - The idea is essentially to cut off the jumps at some level, usually 1, and try to let ε converge to zero properly in

$$X_t = \gamma t + \sigma W_t + \sum_{s \le t; |\Delta_s X| \ge 1} \Delta_s X + X_t^\varepsilon \qquad X_t^\varepsilon = \sum_{s \le t; \varepsilon < |\Delta_s X| < 1} \Delta_s X$$

Levy-Ito decomposition and the characteristic triple (2)

- In fact, this can be achieved by considering the *compensated* process X^{ε} instead if the original one (that's because convergence of martingales can be used).

$$X_t = \gamma t + \sigma W_t + \sum_{s \le t : |\Delta_s X| \ge 1} \Delta_s X + \lim_{\varepsilon \downarrow 0} \left(\sum_{s \le t : \varepsilon \le |\Delta_s X| < 1} \Delta_s X - t \int_{\varepsilon \le |x| < 1} x \nu(dx) \right)$$

This can be done for all Levy processes, and we obtain:

Levy-Ito decomposition and the characteristic triple (3)

- Theorem
  - Each Levy process X can be written as

$$X_t = \gamma t + \sigma W_t + C_t + R_t$$

where

- C is a compound Poisson process with intensity $\lambda := \nu[|x| \geq 1]$ and jump-distribution given by $\mu[A] := \nu[A \cap \{|x| \geq 1\}]/\lambda$.
- R is the limit of *compensated* compound Poisson processes

$$\tilde{X}_t^{\varepsilon} := \left(\sum_{s \leq t} 1_{\varepsilon \leq |\Delta_s X| < 1} \Delta_s X\right) - \nu[[\varepsilon, 1)] \cdot t$$

each of which "contains the jumps of amplitudes in $[\varepsilon, 1)$".

  - This justifies to characterise X by its *characteristic triple* (A, ν, γ).

Characteristic Exponent

- Properties
  - The law μ_t of a Levy process is *infinitely divisible*, i.e. for all n , we can write

$$X_t = \sum_{i=1,\ldots,n} \tilde{X}_i$$

    for some independent $\tilde{X}_i \sim X_{t/n}$.

    - The converse is also true.
    - The law μ_t is the convolution of n copies of $\mu_{t/n}$
  - As a consequence, the characteristic function Φ of X is given as an exponential,

$$\Phi_t(z) := \mathrm{E}[e^{izX_t}] = e^{t\psi(z)}$$

    and we call ψ the *Characteristic Exponent* of X .

    - Brownian motion: $\psi(z) = -{}^1/_2 z^2 \sigma^2 + i\gamma$
    - Poisson-process: $\psi(z) = \lambda\,(e^{iz}-1)$
    - Compensated Poisson-process: $\psi(z) = \lambda\,(e^{iz}-1-iz)$
    - Compensated compound Poisson-process: $\psi(z) = \lambda(\hat{F}(z) - 1 - iz\,\mathrm{E}[Y_1])$

Levy-Khinchin representation

- Theorem
  - A Levy process X with ct (A, ν, γ) has the characteristic exponent

$$\psi(z) = -\tfrac{1}{2} zAz + i\gamma z + \int_{\Re^d} (e^{izx} - 1 - 1_{|x|<1} izx)\, \nu(dx)$$

- This simplifies if the process has *finite activity* (ie if it has a finite number of jumps in any finite period of time).

$$\psi(z) = -\tfrac{1}{2} zAz + i\tilde{\gamma} z + \int_{\Re^d} (e^{izx} - 1)\, \nu(dx)$$

- If $\mathrm{E}[e^{uX_t}] < \infty$, then

$$M_t := e^{uX_t - t\psi(-iu)} = \frac{e^{uX_t}}{\mathrm{E}[e^{uX_t}]}$$

is for all real u a strictly positive, uniformly integrable martingale with unit expectation.

Ito’s Lemma for processes with jumps

- Theorem
  - Of course, Ito’s lemma can be applied to processes with jumps, too. We have

$$\begin{aligned} F(t,X_t)-F(0,X_0) &= \int_0^t \partial_t f(s,X_{s-})ds + \tfrac{1}{2}\int_0^t \partial_{xx}^2 f(s,X_{s-})d\langle X\rangle_s^c \\ &+ \int_0^t \partial_x f(s,X_{s-})dX_s \\ &+ \sum_{s\le t:\Delta_s X\neq 0}\{\Delta_s f(s,X_s)-\partial_x f(s,X_{s-})\Delta_s X\} \end{aligned}$$

where we used $\Delta_s f(s,X_s) := f(s,X_{s-}+\Delta_s X)-f(s,X_{s-})$. If the process X can be written as a sum of a diffusion plus a jump process is of finite variation, we get

$$\begin{aligned} F(t,X_t)-F(0,X_0) &= \int_0^t \partial_t f(s,X_{s-})ds + \tfrac{1}{2}\int_0^t \partial_{xx}^2 f(s,X_{s-})d\langle X\rangle_s^c \\ &+ \int_0^t \partial_x f(s,X_{s-})dX_s^c + \sum_{s\le t:\Delta_s X\neq 0}\Delta_s f(s,X_s) \end{aligned}$$

Ito's Lemma for Levy processes

– By definition of the jump measure, we can restate Ito for Levy processes such that for functions f with $E[|f(X_t)|]$ finite,

$$
\begin{aligned}
F(t,X_t)-F(0,X_0) \quad = \quad & \int_0^t \partial_x f(s,X_{s-})\sigma\, dW_s \\
+ \quad & \int_{[0,t]\times\Re^d} \{f(s,X_{s-}+y)-f(s,X_{s-})\}\,\tilde{J}(d(s,y)) \\
+ \quad & \int_0^t \{\partial_t f(s,X_{s-})+\partial_x f(s,X_{s-})\gamma+\tfrac{1}{2}\partial_{xx}^2 f(s,X_{s-})\sigma^2\}ds \\
+ \quad & \int_0^t \int_{\Re^d} \{f(s,X_{s-}+y)-f(s,X_{s-})-y\partial_x f(s,X_{s-})1_{|y|<1}\}\nu(dy)ds
\end{aligned}
$$

– Using $f(s,x):=e^x$, the process e^X is seen to be a martingale if $\mathrm{E}[e^{X_t}]<\infty$ and

$$\gamma = -\tfrac{1}{2}\sigma^2 - \int_{\Re^d}\{e^y - 1 - y1_{|y|<1}\}\nu(dx)$$

- Using Levy processes to model the stock price process

Modelling the stock price

- Levy processes can be used to model the returns of a stock, ie the stock S with forward curve F_t is assumed to be given as

$$S_t = F_t \frac{e^{X_t}}{\mathrm{E}[e^{X_t}]}$$

for some Levy process X (we assume $\mathrm{E}[e^{X_t}] < \infty$).
  - Markov-Process with independent increments.
  - Stationary distribution.
  - Large jumps for extreme market movements.
  - Small jumps and diffusion for instantaneous trading.

- It is obvious that $X_t = \sigma W_t$ is a special case.
  - However, *time-dependent* volatility is *not* (yet) contained in our model class.

Examples: Merton-Model (1)

- Merton's model (1976): Black-Scholes plus Jumps

$$X_t = \mu t + \sigma W_t + \sum_{i=1,\dots,N_t} Y_i$$

  - Usually, Y_i are Gaussian $N(m,v)$.
  - The characteristic exponent is $\psi(z)=-{}^1/_2\, z^2\sigma^2+iz\mu+\lambda\,(\exp(-{}^1/_2\, v^2z^2+izm)-1)$
    - Note that the last bit is just the characteristic function of the Gaussian jumps.
  - To obtain a martingale, use

$$\mathrm{E}[e^{X_t}] = e^{(\mu+\frac{1}{2}\sigma^2)t}\sum_n \mathrm{P}[N_t = n](e^{m+\frac{1}{2}v^2})^n = e^{\{\mu+\frac{1}{2}\sigma^2+\lambda(e^{m+\frac{1}{2}v^2}-1)\}t}$$

  - European Option can be priced easily with a series development.
  - Kou model: Asymmetric exponential.

Examples: Merton-Model (2)

- SDE of the model: Using our Ito-formula on $S_t = e^{X_t}$ yields

$$\begin{aligned} dS_t &= S_t dX_t^c + S_t d\langle X\rangle_t + d\left(\sum_{s\le t;\Delta_s X\neq 0} e^{X_{s-}+\Delta_s X} - e^{X_{s-}}\right) \\ &= S_t\left\{(\gamma - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t + (e^{\Delta_t X} - 1)\right\} \\ &= S_t\left\{(\gamma - \tfrac{1}{2}\sigma^2)dt + \sigma dW_t + dP_t\right\} \end{aligned}$$

  – The process P is *another* compound Poisson process

$$P_t = \sum_{i=1,\dots,N_t} (e^{Y_i} - 1)$$

"Merton-Model" for Hybrid Models with Default Risk

- Now take Merton's model and let m go to negative infinity.
  - The return at the first jump will be -100%: *Default.*

$$S_t = S_0 \exp\{\sigma W_t + (r + \lambda - \tfrac{1}{2}\sigma^2)t\} 1_{N_t = 0} = \hat{S}_t e^{\lambda t} 1_{N_t = 0}$$

  - Observing that

$$\begin{aligned} \mathrm{E}[(K - S_t)^+] &= \mathrm{E}[1_{N_t=0}(K - \hat{S}_t e^{\lambda t})^+] + \mathrm{P}[N_t > 0]K \\ &= \mathrm{E}[(e^{-\lambda t}K - \hat{S}_t)^+] + (1 - e^{-\lambda t})K \end{aligned}$$

    we can use the Black&Scholes-formula to compute put prices.
  - Can be used to model the default of the asset. In this model, the default is not related to the stock movement.
  - The latter drawback can be tackled by using a "stochastic intensity" process λ instead of a deterministic number resp. function.
    - We line this out in the "extending the Levy processes" section in the end.

Subordinators

- <u>Definition</u>
  - A Levy process Z is called a *subordinator*, if it has a.s. non-decreasing paths.
    - Obviously, such a process cannot have a diffusion component.
  - Given a Levy process X and an independent subordinator Z, the process

$$Y_t = X_{Z_t}$$

is called a *subordinated* Levy process. If l is the Laplace-exponent of the subordinator, and ψ the characteristic function of X, then the characteristic exponent of Y is given by

$$l(\psi(z))$$

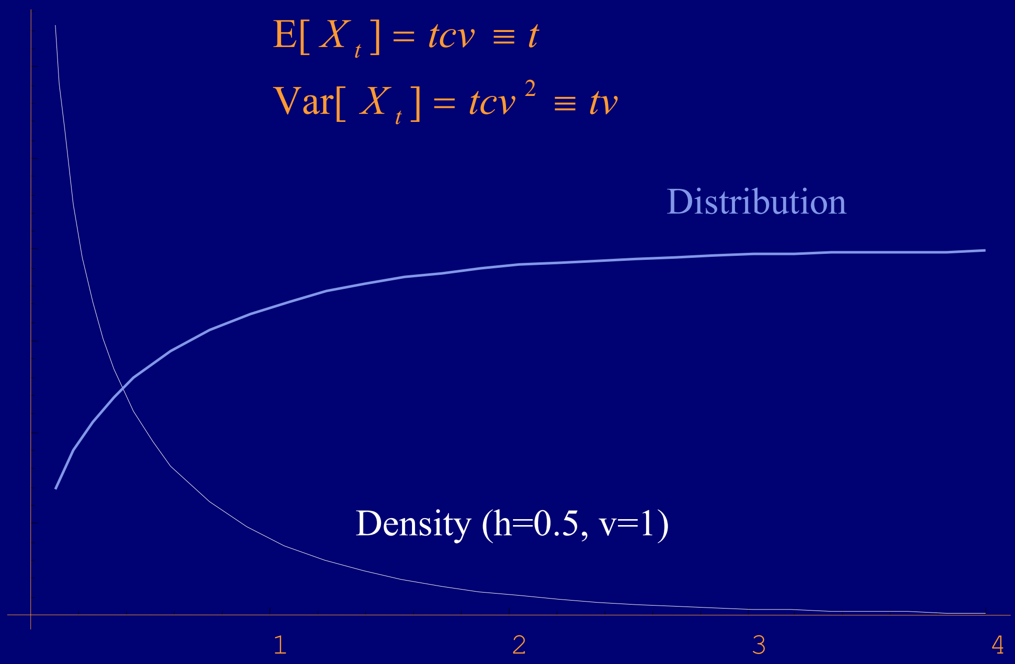
Examples: Variance Gamma (1)

- Variance Gamma (1990)
  - A *Gamma process* is a Levy process where the increments have Gamma distribution

$$P[X_t \le x] = \int_0^x \frac{e^{-y/v} y^{ct-1}}{v^{ct}\Gamma(ct)} dy \overset{c:=1/v}{=} \int_0^x e^{-y/v} y^{t/v-1} v^{-t/v} / \Gamma(t/v) dy$$

$$E[X_t] = tcv \equiv t$$

$$\mathrm{Var}[X_t] = tcv^2 \equiv tv$$



Examples: Variance Gamma (2)

- The Laplace-Transform is

$$E[e^{u\gamma_t}] = (1-uv)^{-t/v} = e^{-1/v \log(1-uv)\, t}$$

- The *Variance Gamma* process is the result of a time-changed Brownian motion with drift b and volatility σ using a Gamma-process γ with variance v:

$$X_t = \mu\gamma_t + \sigma W\gamma_t$$

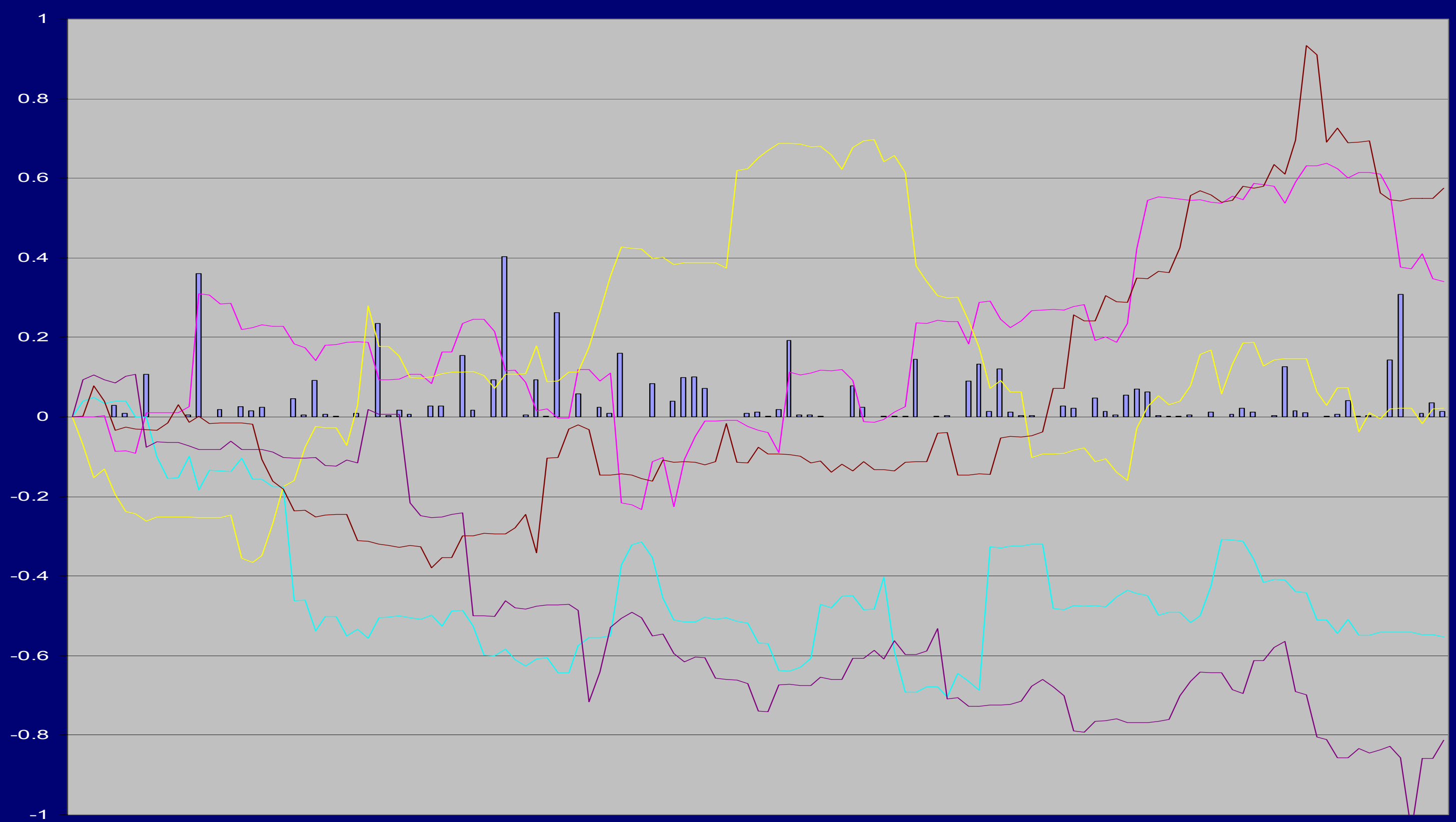
- The characteristic exponent is

$$\psi(z) := -\tfrac{1}{v}\log(1 + \tfrac{1}{2} z^2\sigma^2 v - izbv)$$

  - The constant v determines the variance of the sub-ordinator at time 1.
  - Note that the variance of the process conditional on γ is γ itself.
  - VG has finite variation, but infinite activity.

Examples: Variance Gamma (3)

– Some VG sample paths and the gamma samples (b=0.3, σ=30%, v=40%^{2})



Examples: CGMY

- CGMY (Carr, German, Madan, Yor 2000)
  - An extension of the VG process is

$$X_t = \mu h_t + \sigma W_{h_t}$$

where h is a generalized form of the VG process. The characteristic exponent is

$$\psi(z) := C\Gamma(-Y)\left\{(M - iu)^Y - M^Y + (G + iu)^Y - G^Y\right\}$$

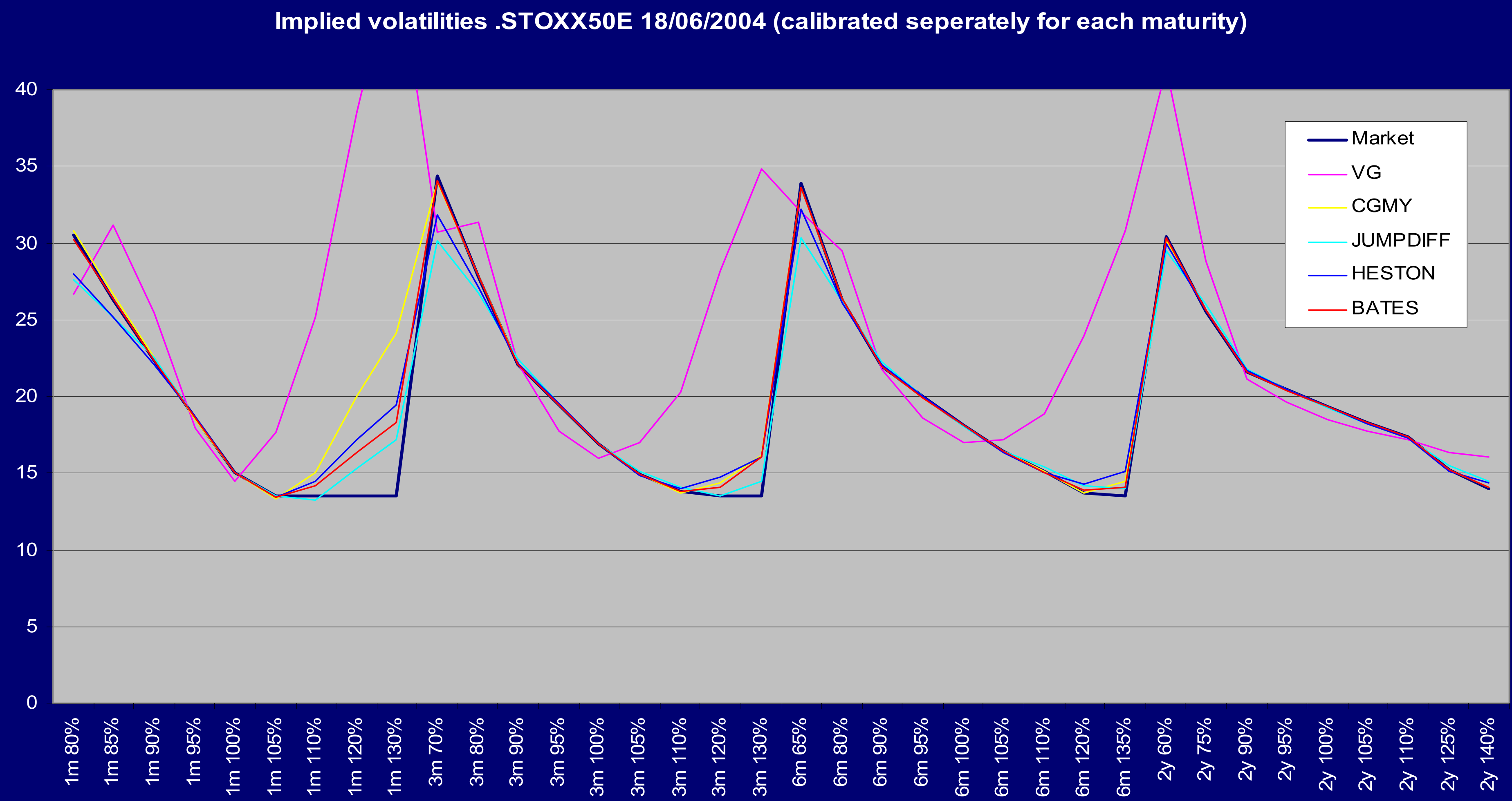
- C, G, M, Y must be positive.
  - $-1<Y<0$: Compound Poisson process
    $0<Y<1$: Infinite activity, but finite variation
    $1<Y<2$: Infinite activity and finite quadratic variation
- With the above limitations, the process is a time-changed BM with drift.
- This process produces very nice shapes for implied volatility smiles at a fixed maturity.

  - We can easily add a Brownian component (CGMYe).

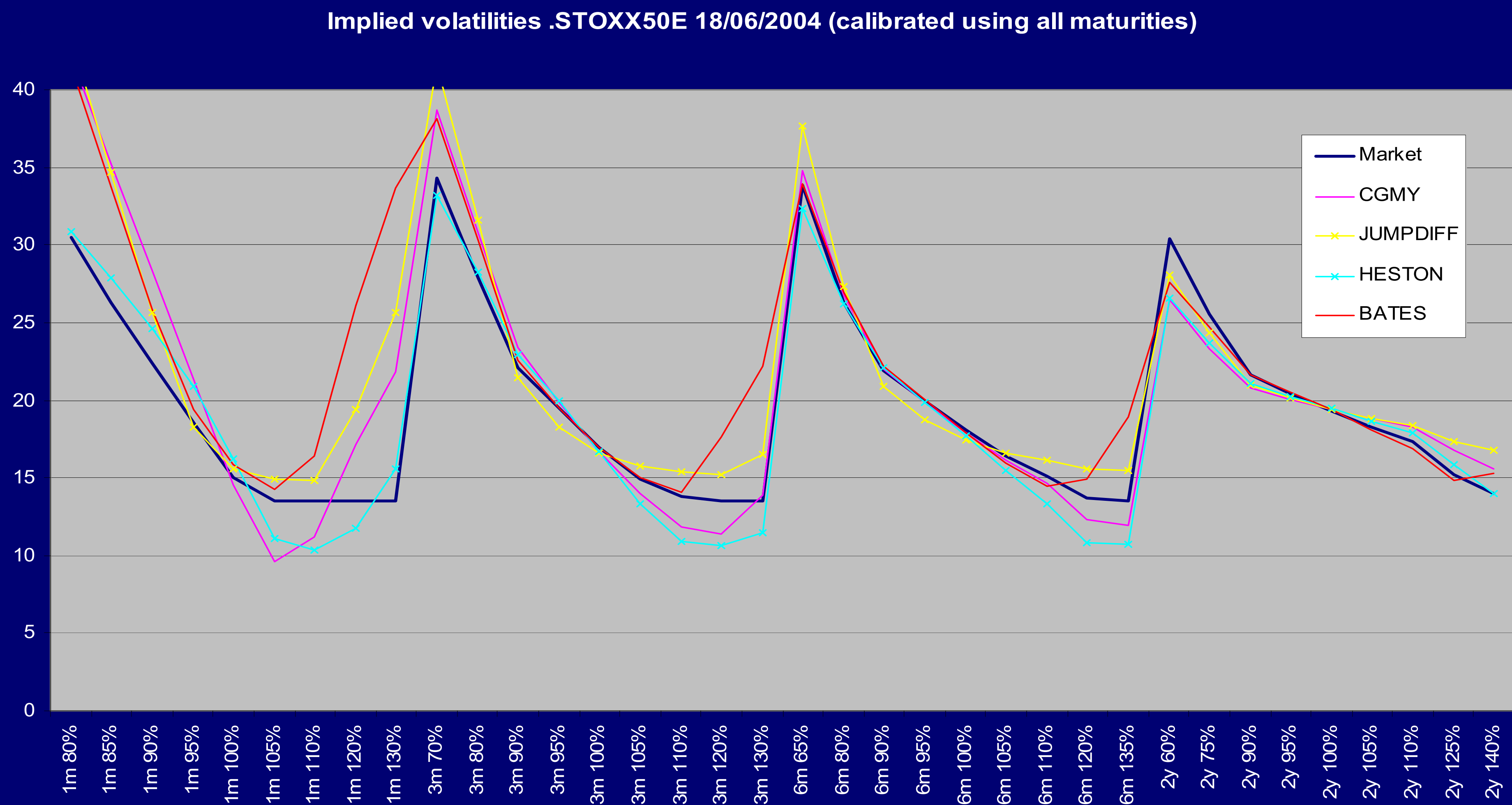
Pricing Vanillas a'la Carr/Madan

- To compute the price of a European option with payoff H, we can in principle invert the characteristic function of X.
  - However, we want an efficient way to do so.
- We need a fast algorithm for calibration: Carr & Madan proposed an FFT algorithm to improve the performance of the European Call price.
  - See talk from Tistaert.

Implied Volatility of calibrated Levy processes (1)



Implied Volatility of calibrated Levy processes (2)



- Numerical Methods

Numerical procedures – Monte-Carlo (1)

- How to simulate a Monte-Carlo process for a Levy process.
  - Simulating a Merton-type process
    - (A) If we we have a payoff defined on some fixing dates
      - Between two dates, first obtain the number of jumps.
      - We need the sum of the jumps. For Gaussian jumps, this is *one* Gaussian.
      - Now the process is just a geometric BM with drift.
    - (B) If hitting times of some barriers are involved
      - First compute the number n of jumps.
      - Compute the jump times - conditional on the number of jumps and the length of the interval, the jump times are given by n uniform variables in $[0,T]$.
      - Simulate the jumps themselves.
      - Brownian motion in between, use for example Brownian Bridge to refine result locally for barriers (cf. GL00)

Numerical procedures – Monte-Carlo (2)

– Simulating subordinated Brownian motion (cf.. CT04)

- Simulate the subordinator (for VG this is is a Gamma-process and can be found in CT04 or GL00), then the Brownian motion.

– For some processes, there is no "easy" way of performing the simulation. Recall

$$X_t = \gamma t + \sigma W_t + \sum_{s\le t:|\Delta_s X|\ge 1} \Delta_s X + \lim_{\varepsilon\downarrow 0}\left(\sum_{s\le t:\varepsilon\le|\Delta_s X|<1}\Delta_s X - t\int_{\varepsilon\le|x|<1} x\nu(dx)\right)$$

- First idea: Approximate small jumps by expectation. Works if their variation is small:

$$X_t^{(\varepsilon)} = \tilde{\gamma} t + \sigma W_t + \sum_{s\le t:|\Delta_s X|\ge\varepsilon}\Delta_s X + t\int_{|x|<\varepsilon} x\nu(dx)$$

- If not, the error has zero expectation but variance $\sigma^2(\varepsilon) := \int_{|x|\le\varepsilon} x^2\nu(dx)$ and converges against a process without jumps. Hence, use

$$\hat{X}_t^{(\varepsilon)} = \hat{\gamma} t + (\sigma+\sigma(\varepsilon))W_t + \sum_{s\le t:|\Delta_s X|\ge\varepsilon}\Delta_s X + t\int_{|x|<\varepsilon} x\nu(dx)$$

Numerical procedures – Finite Differences (1)

- In the Gaussian case, the price

$$V(t, X_t) = e^{-r(T-t)}\,\mathrm{E}\big[H(S_T) \,|\, X_t\big]$$

of a vanilla option satisfies according to Feynman-Kac

$$0 = \partial_t V + L^* V - rV \qquad L^* = (r - \tfrac{1}{2}\sigma^2)\partial_x + \tfrac{1}{2}\sigma^2 \partial^2_{xx}$$

(we consider all payoffs as functions of the log of the stock).

- In the case of a Levy process, set $r=0$ and let

$$S_t = \exp\left\{\gamma t + \sigma W_t + \int_{[0,t]\times\Re} x\,\tilde{\mathrm{J}}[d(s,x)]\right\}$$

with the deterministic drift

$$\gamma = -\tfrac{1}{2}\sigma - \int_{\Re} (e^x - 1 - x 1_{|x|\le 1})\,\nu(dx)$$

Numerical procedures – Finite Differences (2)

– Using the Ito-formula for Levy processes (a couple of slides back), we see similarly to the Gaussian case

$$0 = \partial_t V + LV$$

but with an *integro-differential* operator

$$\begin{aligned} L\,f &:= \gamma\partial_x f(t,x) + \tfrac{1}{2}\sigma^2 \partial_{xx}^2 f(t,x) \\ &+ \int_{-\infty}^{\infty} \left(f(t,x+y) - f(t,x) - y 1_{|y|\le 1}\, \partial_x f(t,x) \right) \nu(dy) \\ &= \tfrac{1}{2}\sigma^2 \left(\partial_{xx}^2 f(t,x) - \partial_x f(t,x) \right) \\ &+ \int_{-\infty}^{\infty} \left(f(t,x+y) - f(t,x) - (e^y - 1)\partial_x f(t,x) \right) \nu(dy) \end{aligned}$$

Numerical procedures – Finite Differences (3)

– Hence,

$$Lf = L^* f + \int_{-\infty}^{\infty} \Big(f(t, x+y) - f(t,x) - (e^y - 1)\partial_x f(t,x) \Big) \nu(dy)$$

- Problem: The integral term is non-local.
- In contrast to a local operator, boundaries play a bigger role (since we are going to integrate over them with ν). This is no problem for double-barriers (values beyond boundaries clear), but makes vanillas subject to boundary-approximation.
- The Levy measure ν should allow for appropriate functions, ie $\int_{|x|\geq 1} |x|^p \, \nu(dx) < \infty$ for some p larger than 2 (to ensure that the integral exists for some reasonable f). Also assume that the second moment of the price process exists.

Numerical procedures – Finite Differences (4)

- Multi-nomial tree with moment-matching.
  - Moments are available via characteristic function
  - To capture large jumps, the tree must branch widely.
  - Explicit scheme, but martingale property respected by construction.
- Finite Differences
  - Use Brownian approximation as discussed for the Monte-Carlo, if necessary. This reduces the problem to the case of finite activity.
  - Then we have to limit the domain of the integral of ν (may require extrapolation of the value function).
  - We have a brief look at this strategy.
    More details can be found in CT04, pg..412ff.

Numerical procedures – Finite Differences (5)

– Step 1: Reduce to finite activity case $\nu[\Re] = \lambda < \infty$. The integral reduces to

$$\int_{-\infty}^{\infty} \left(f(t, x+y) - f(t,x) - (e^y - 1)\partial_x f(t,x) \right) \nu(dy)$$

$$= -\lambda f(t,x) - \alpha \partial_x f(t,x) + \int_{-\infty}^{\infty} f(t, x+y) \nu(dy)$$

such that our operator has the form

$$Lf \approx af + b\partial_x f + c\partial_x^2 f + \int_{-\infty}^{\infty} f(t, x+y) \nu(dy)$$

Numerical procedures – Finite Differences (6)

– Step 2: Limiting the boundary of the Levy-integral.

$$Lf \approx af + b\partial_x f + c\partial_x^2 f + \int_D^U f(t, x+y)v(dy)$$

- Expression can already be processed when the state/time space is discretizised.

– Step 3: Discretization of the operator

- Denote by D the Discretization of the differential operator and by J the integro-operator. We get

$$\frac{u^{n+1} - u^n}{\Delta t} = (D+J)u^n$$

Explicit scheme with the usual constraints on stability

$$u^{n+1} = (E_n + \Delta t(D+J))\,u^n$$

Numerical procedures – Finite Differences (7)

– Implicit scheme not suitable, since the matrix J is dense.

Hence, use implicit for L and explicit for J.

$$\frac{u^{n+1}-u^{n}}{\Delta t}=(\theta\, D+J)\, u^{n}+(1-\theta)\, Du^{n+1}$$

Ie

$$(E_{n}-(1-\theta)D)\, u^{n+1}=(E_{n}+\Delta t(\theta\, D+J))\, u^{n}$$

Pricing Barriers (1)

- Pricing barrier options - some theoretical ideas
  - Let q>0. Then there existsthe unique Wiener-Hopf factors Φ^+ and Φ^- such that

$$\frac{q}{q-\psi(z)} = \Phi^+(z)\,\Phi^+(z)$$

Now let h be the joint characteristic function of $(M_t, X_t - M_t)$, with $M_t := \sup_{s \leq t} X_s$ then

$$q\int_0^\infty e^{-qt} h_t(x,y)dt = \Phi^+(x)\Phi^-(y)$$

  - Denote by $UIC(t,k,b)$ the value of an up-and-in call with maturity t, log-strike k and log-barrier b.

$$q\int_{\Re^2} e^{iuk+ivb}\int_0^\infty e^{-qT} UIC_\alpha(T,k,b)dTdkdb = \frac{\Phi^+(u+v-i)\Phi^-(u-i)}{-uv(iu+1)}$$

  - This gives us a theoretical result, but the integral is actually quite complicated to compute. One is better off with Monte-Carlo.

Pricing Barriers (2)

- Now assume we have only downward jumps
  - Here are some pseudo-closed formulas if the process has only downward jumps:
    - First, it is clear that X has the value x at the stopping time $\tau := \inf\{t: X_t > x\}$.
    - Using the standard arguments, we can therefore conclude

$$1 = \mathrm{E}[e^{zX_\tau - \tau\psi(iz)} 1_{\tau<\infty}] = \mathrm{E}[e^{zx - \tau\psi(iz)} 1_{\tau<\infty}]$$

    - In the case of no positive jumps, $l(z) := \psi(iz)$ is decreasing and can be inverted. This yields the Laplace-transform of τ:

$$\mathrm{E}[e^{-\tau u} 1_{\tau<\infty}] = e^{-x l^{-1}(u)}$$

    - Now let h be the joint characteristic function of $(M_t, X_t - M_t)$, with $M_t := \sup_{s\le t} X_s$ then

$$\begin{aligned} UIC(t,k,b) &= \mathrm{E}[1_{\tau\le T}(e^{X_T} - e^k)^+] = \mathrm{E}[1_{\tau\le T} e^x (e^{X_T - X_\tau} - e^{k-x})^+] \\ &= \mathrm{E}[1_{\tau\le T} \mathrm{C}(T-\tau, k-x)] = e^x \int_0^T \mathrm{C}(t, k-x) dF^\tau(t) \end{aligned}$$

where the latter integral with respect to the distribution F^τ of τ can be computed using Laplace-transforms.

The impact of a stationary distribution: Forward Starts

- Using a stationary distribution implies similar prices for the contracts of the same "period".
  - If we disregard Forwards and Discountfactors, an Option on the increment of length τ at some starting time t will have the same price as if started today.
  - Forward-Start options for example, *should* somehow have this behaviour:

$$(S_T / S_t - k)^+ \qquad (S_T - kS_t)^+$$

    - Note that if S is a Levy-process, both options have the same price.
    - The price is also the same for all options with the same $\tau = T - t$.
    - This is in principle a desirable feature, but the ATM options don't reflect today's prices.

- A comment on hedging with Levy processes

Hedging in Levy models (1)

- We have a price - and now?
  - First problem: Parameters for our model may not be stable over time.
  - Second problem: Jump processes create are inherently incomplete markets.

- "Vega"-Hedging
  - In Black&Scholes, we actually have the same "first" problem: *Volatility* is a parameter for the model and may change.
    Since any of our Levy models will also depend on some parameter θ, we should eliminate first order sensitivity to these parameters.
  - Works reasonably if the parameters do not change too much.
  - To do it rigidly, the concept of "uncertain parameters" might be applied.
    However, this is numerical involved (for the geometric BM case, an application of this is outlined in WL98).

Hedging in Levy models (2)

- Incomplete markets
  - Even if our parameters fit the observed options well, there might be more than one martingale-measure.
    - In theory, we should model the stock in “real life”. However, this is somehow inpractical (note that the set only very few models classes are closed upon equivalent changes of measures) .
    - But what does the existence of traded options imply?
  - Superhedging is too expensive.
    - Indeed, for Levy processes, Bellamy/Jeanblanc showed that for a call, the price bounds are given by the pure Black&Scholes price on one hand, and the pure stock price on the other.

Hedging in Levy models (3)

- Mean-Variance hedging is a clear alternative.
  - The idea is to minimize the variation of the payoff from the hedge.
    - It treats profit and loss equally.
  - Closed form available cf. BL89 or CT04.
    We basically have an adjusted delta. It reduces to BS-Delta if no jumps are present.
  - Concept appealing, but it does not capture the existence of traded options.
    - How incomplete is a market with a variety of traded options?
    - This is theoretically not properly solved, yet.

- Extending Levy processes

Where are we?

- Advantages of Levy processes
  - Nice mathematical features.
  - Good fit to smile at one maturity for CGMY. More freedom available.
  - Implied volatility flattens out as expected.
  - Non-deterministic variance despite simple structure.
  - Stationary distribution.
- Drawbacks
  - Fits for more than one maturity not great.
  - Stationarity of distribution too rigid.
  - Numerics can be quite involved.
  - No-memory-property is not a good model of “reality”.

Using Levy models as components (1)

- We can still rely on the relative tractability of Levy processes to combine them with other processes
  - Early example: Bates' model
    - Stochastic (Heston) volatility plus jumps in the stock
    - Can also be used to add jumps to the volatility itself.
    - The jumps are normally compound poisson processes.
  - In particular jumps are quite easy to handle and can capture other market effects such as defaults, sudden movements, switches in volatility etc.
  - Using a stochastic intensity we can model some dependence between the Brownian motion of the stock and the jumps.

Using Levy models as components (2)

- Let Λ be an increasing adapted functional of a Brownian motion W, and define a compound Poisson process this intensity Λ. The jumps Y are assumed to be independent with characteristic function F and mean m. Let $B=\rho W+\rho' W^2$ and consider

$$S_t = \exp\left\{ \int_0^t \sigma dB_s - \tfrac{1}{2}\int_0^t \sigma_s^2 ds + \sum_{i=1,\ldots,N_t} Y_i - \Lambda_t m \right\}$$

- We want to get the characteristic function of $\ln S$.
  - First split $B=\rho W+\rho' W^2$ and condition on W^2. Also remove the drift.
  - Condition on W. This yields some deterministic function a such that

$$\mathrm{E}\left[\exp\left\{ \int_0^t a_s dW_s - iz\Lambda_t (F(z)-m) \right\}\right]$$

  - Use Girsanov to remove the first part; under the new measure, W changes as usual. Under the new measure, compute the characteristic function for Λ:

$$\widetilde{\mathrm{E}}\left[\exp\left\{ iz\Lambda_t (F(z)-m)\right\}\right] = \widetilde{\Phi}(z(F(z)-m))$$

Definition: Additive processes

- <u>Definition</u>
  - A cadlag process $X = (X_t)_{t \geq 0}$ is called a *Additive process* iff
    - it has independent increments and if
    - it is stochastic continuous.
  - Idea: Make parameters of the model time-dependent.
  - When does this work?

Additive processes - Sato's theorem

- Theorem
  - Any additive process X has an infinitely divisible distribution, and it is determined by its deterministic *spot characteristics* $(A_t, \nu_t, \gamma_t)_t$. Its characteristic function is

$$E[e^{izX_t}] = e^{\psi_t(z)}$$

with *characteristic exponent*

$$\psi_t(z) = -\tfrac{1}{2} zA_t z + i\gamma_t z + \int_{\Re^d} (e^{izx} - 1 - 1_{|x|\leq 1} ixz)\, \nu_t(dx)$$

- For all $t>s$, $A_t - A_s$ is positive definite, $\nu_t - \nu_s$ must be a positive and appropriately integrable.
- All components must be continuous, the measures must converge outside the origin. [see CT04 pg.. 457 for details]

Additive processes - Sato's theorem

– A convenient way to construct such a process is by using

$$A_t = \int_0^t \sigma_s^2 ds$$

$$v_t = \int_0^t \mu_s ds$$

$$\gamma_t = \int_0^t g_s ds$$

- The function σ must be square-integrable, g must be integrable and
- the family μ needs to fulfil

$$\int_0^T \int_{\Re^d} (|x|^2 \wedge 1)\, \mu_t(dx)dt < \infty$$

- The triplet (σ, μ, g) is called the *local characteristic* of X.

Additive processes - Sato's theorem

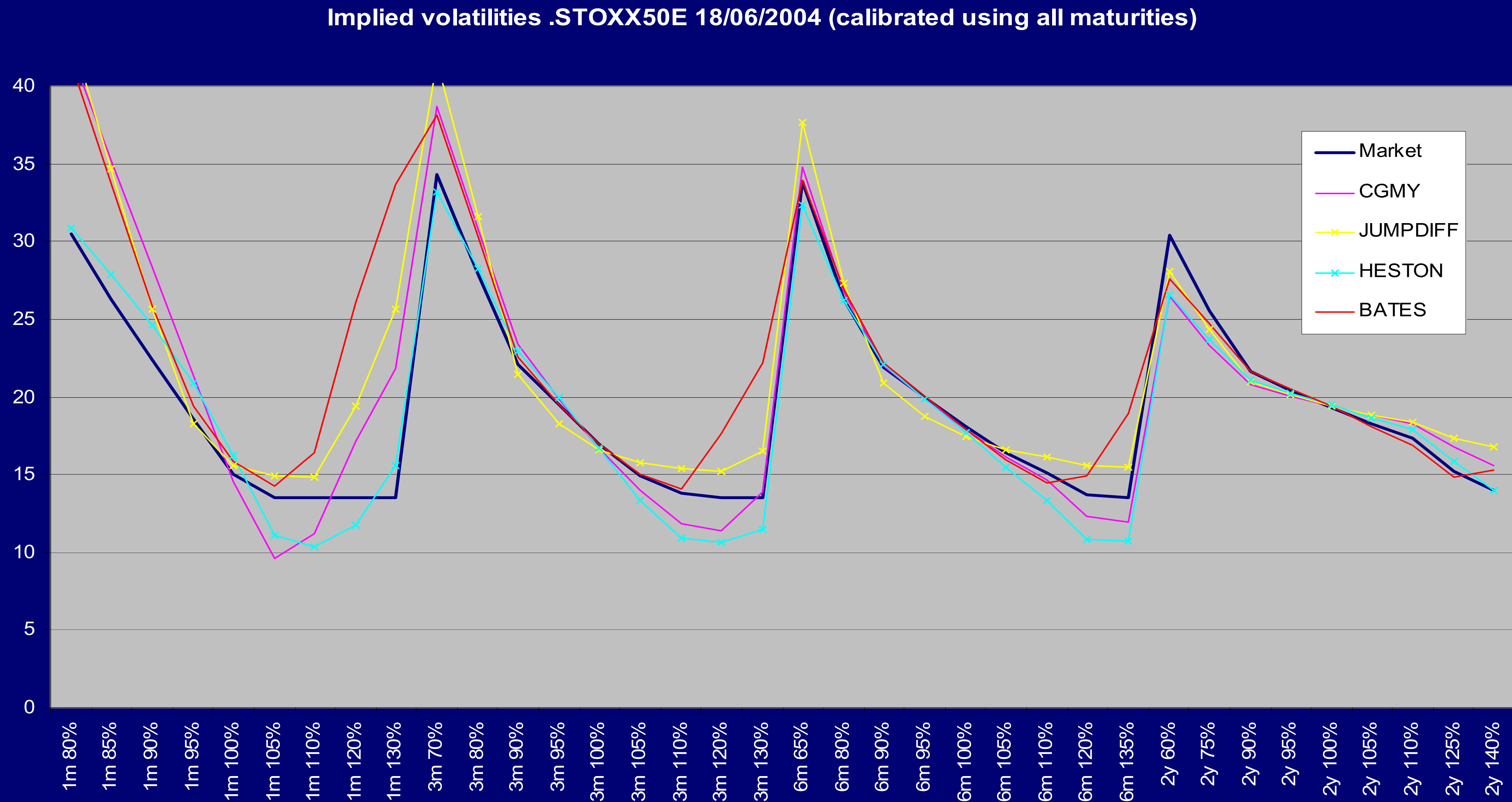
- The main observation here is that for each fixed t, we can rewrite the characteristic exponent as

$$\psi_t(z)/t = -\tfrac{1}{2}z(A_t/t)z + i(\gamma_t/t)z + \int_{\Re^d}(e^{izx} - 1 - 1_{|x|\le 1}ixz)(\nu_t/t)(dx)$$

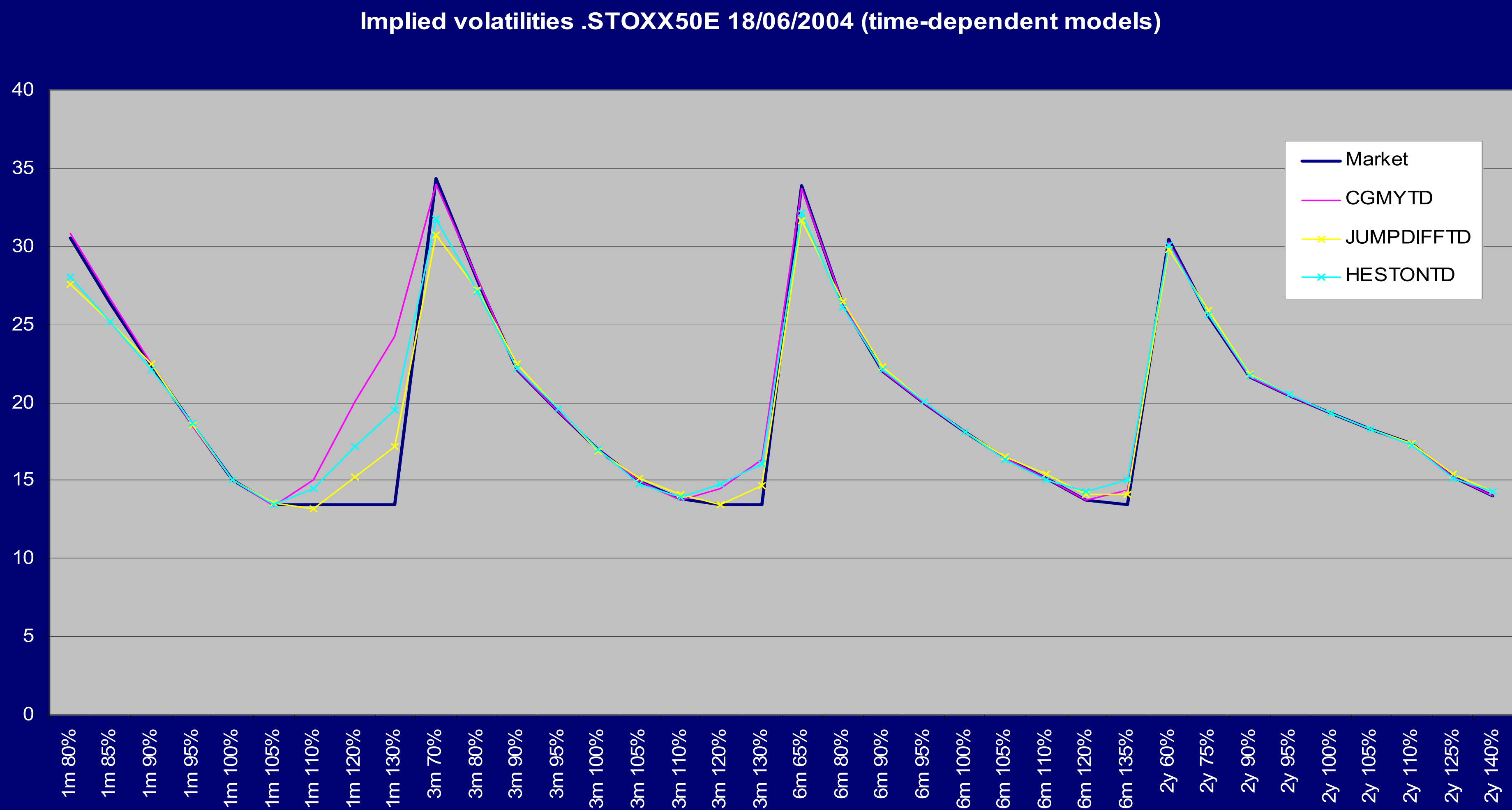
  hence the process in t can be treated as a normal Levy process with characteristic triple $(A_t/t, \nu_t/t, \gamma_t/t)$.

- European options can be priced with the same algorithm as before.
- Simulations are equally straight-forward at least if the local parameters are piecewise constant.

Implied Volatilities (recall)



Implied Volatilities for time-dependent parameters



- Thank you for your attention.

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- Literature
  - Most of the material presented can be found in one or the other form in the highly recommended book

    CT04: Cont/Tankov: “Financial Modelling with Jump Processes” (2004)

  - Other books
    - WM98: Wilmot “Quantitative Finance” (1998) - practical view point.
    - OV02: Overhaus et al “Equity deriviatives” (2002) - quite dense introduction with most topics covered, including barriers.
    - FS02: Foellmer/Schied “Stochastic Finance” (2002) - theory for incomplete markets.
    - GL00: Glassermann “Monte-Carlo Methods in Financial Engineering” (2000) - as the title suggests, a comprehensive guide into Monte-Carlo

- Papers
  - Bates, 1996: *Jumps and stochastic volatility: exchange rate process implicit in DM options*. Rev. Fin. Studies 9-1
  - Bouleau et al, 1989: Residual risks and hedging strategies in markovian markets. Stochastic Process. Appl., 33 (1989), pp. 131-159
  - Carr et al, 1998: *Option Valuation Using the Fast Fourier Transform.* Journal of Computational Finance, 2, pp. 61-73
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