Construction of Martingales Under Constraints:

From Implied Volatility to Pricing Exotics

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Tandem-Workshop Stochastik-Numerik
Berlin, 11.6.2004

Deutsche Bank

DFG research center Berlin
mathematics for key technologies

Outline

- Introduction: The Project and its Purpose.
- Hard and Soft Market Information
- Implied Volatility and Absence of Arbitrage
- Construction of Martingales:
  - Discrete Case
  - Continuous States
  - Smooth surfaces and Diffusions
- Other approaches
- Hedging and Dynamics

Introduction

- “Construction of Martingales Under Constraints”
  - The Project is a cooperation between TU Berlin (Prof. Schied) and Deutsche Bank London, Global Equities Quantitative Research.
  - It is aimed at the construction of pricing systems which are consistent with both “hard” and “soft” market information.
  - We aim to
    - Understand static and dynamic information
    - Assess the theoretical implications of market information
    - Construct consistent pricing systems for application in live trading
  - The following talk will give an overview of the the state and problems of the project.

Hard Information...

- The Market trades not only the Share S itself, but also (a finite number of) European options.
  - An European *"Call"* with maturity T and strike K pays

No dividends, no interest rates and $S_0=1$

$$(S_T - K)^+$$

- These options are usually quoted in *Implied volatility* $\sigma(T,K)$
  - Computed by inverting the B&S formula for the call price

$$C(T,K) \equiv BS(T,K,\sigma(T,K)\sqrt{T})$$

$$BS(\tau,\kappa,\Sigma) := \mathrm{N}(d^+) - \kappa\,\mathrm{N}(d^-)$$

$$d^{\pm} := -\ln\kappa/\Sigma \pm \tfrac{1}{2}\Sigma$$

NB: Nobody assumes that B&S actually holds.

... and Soft Information

- The prices of some other Structures are *approximately* known
  - Example: A “Floating-Strike Forward Start Call” from t to T pays

$$(S_T - kS_t)^+$$

The trading has a means to evaluate this option by using the B&S formula with an appropriately adjusted volatility.
Assumption for example: The ATM options ($k=1$) *are* B&S priced.

- Such information is not necessarily correct, but should be taken as a strong indication of the “true price”
  - The “right” price is set by the market, not a model – until we find a hedge.
- What means a price without a *Hedging (replication) Strategy*?
  - Can we at least superhedge our risk exposure?

Construction of Martingales Under Constraints

- The aim is to find “pricing systems”, i.e. martingale measures, which can explain both hard and soft information.

- We also have to focus on the dynamical behaviour of any model we propose - if it diverges from the market, its implied hedging strategies will as well.
  - Plain super-hedging too expensive.
  - Alternative risk-based hedging strategies give bounds.

Absence of Arbitrage

- As usual: If the market is arbitrage-free, there exists at least one martingale measure for all traded instruments.
- Given such a measure, $C(T,K)=\mathrm{E}[(S_T - K)^+]$ is
  - bounded from below by $(1-K)^+$, from above by 1,
  - strictly decreasing with limit zero, convex in strike and
  - increasing in time.
- Reversely
  - For each T, $\mu_t[-\infty, K] := 1 + \partial_K^- C(T,K)$ can be constructed.
  - Kellerer (1972) has shown that there is a Markov-martingale with these marginals if the distributions μ_t are ordered in the Balayage-order, i.e. if the Call-Prices are increasing in time for each K.

Construction of Martingales - Discrete Case (1/2)

- We need to check whether the surface is arbitrage-free if we want to fit a martingale to it.
  - Marginals can be constructed if basically
    - First finite differences must be between -1 and 0
    - Second FDs must be positive
  - Monotonicity in time is trivial if strikes the same for all maturities (*rectangular grid)*, but a bit more involved otherwise - for each single maturity, use convex hull of later maturities as reference.
- We need to *produce* an Arbitrage-free surface if we want to create a discrete state Markov-process.
  - LP resp. LSQ problem.
    Easy for rectangular grid, but slow for the general case (which is the common situation due to Dividends, Interest rates etc).

Construction of Martingales - Discrete Case (2/2)

- We end up with a set of state/probability pairs $(x^i,p^i)_i$.
- Find transition matrices K^i by solving
  - Kernel-propery: $K^i 1=1$ with K positive and $p^{i-1} K^i = p^i$
  - Martingale-property: $K^i x^{i-1} = x^i$
  - Weak information enter as optimality criteria, and are typically linear in the coefficients of K.
- Works.
  - Monte-Carlo very quick ($\rightarrow$ control variates in discrete state MC?)
- Technical Problems
  - For small number of strikes only of theoretical appeal.
  - For large number of strikes quite slow (at least with the straight-forward use of NAG 's LP/LSQ algorithm).

Construction of Martingales - Continuous States (1/3)

- Problem:
  - Construction of a discrete number of smooth distribution functions from discrete data under No-Arbitrage Constraints for each maturity.
  - Once successful, the density of the stock at each maturity can be computed from the interpolation. European options can be priced.

Construction of Martingales - Continuous States (2/3)

- Approach 1 - Interpolation in Strike:
  Linearly constrained cubic splines in strike (LP/LSQ problem)

  – Calls are not Cubic.
  – No-Arbitrage conditions are LP.
  – Severe constraints on the bounds are numerically challenging, in particular if we require strictly positive second derivatives (the functions have to be increasing in time, but must not exceed some “maximal volatility” boundary).
    $\rightarrow$ Short maturities must be excluded.

Construction of Martingales - Continuous States (3/3)

- Approach 2 - Interpolation of the Implied SqrtVar Σ surface: Constrained cubic splines in strike or log-strike.
  - Log-strike version is more natural (market often operates with parametric forms in log-strike).
  - Global optimization (all maturities together) slow. Blockwise bootstrap.
  - Severe problems in imposing No-Arbitrage conditions.
    Heuristic: Convex volatility.
    Expensive post-processing necessary…?

Construction of Martingales - Smooth Surfaces and Transition Kernels (1/3)

- Once the marginals are given, interpolate in time.
  - Market standards: Linear in Variance or SqrtVar.
  - Both Call and SqrtVar-Surfaces yield a „Local Volatility“ if the interpolation in strike is smooth enough

$$d\sigma(t,x)^2 = \frac{\partial_T C(t,x)}{2\partial^2_{KK} C(t,x)}$$

    - If the Call prices are strictly convex, the local volatility coefficient exists.
    - Smoothness of the function not guaranteed, but speeds up solving the SDE

$$dS_t = \sigma(t,S_t)dW_t$$

  - Local volatility is theoretical very nice but must be used with the utmost care in practise. Forward-Starts and options on variance are severely mispriced (*this is a piece of weak information...*).

Construction of Martingales - Smooth Surfaces and Transition Kernels (2/3)

- Advantages of this method:
  - Marginals for *all* maturities are available for immediate quality checks (and control variate techniques in numerical schemes).
  - Extrapolation part of the modelling and can be adjusted to users.
  - Transformations of the surface can be implemented as surface transformations (for example for risk management)
  - Not bound to some numerical scheme.
- Disadvantages
  - Local volatility hard to control in terms of smoothness (which is easier in numerical schemes).
  - Usual drawback: Non-locality of splines.

Construction of Martingales - Smooth Surfaces and Transition Kernels (3/3)

- *But in fact, we want to incorporate soft information.*
  $\rightarrow$ Open problem.

- Construction of smooth transition kernels from marginals under constraints.
  - Can Wavelet-bases be used?
  - Analytic approaches?

Other approaches

- Statistics on the implied volatility surface to determine its behaviour
  - Rama Cont et al (2002) - Analysis of S&P 500 implieds
    - Surface movements can be explained using OU processes.
    - Leads itself to an application of “stochastic implied volatility”
  - Härdle et al (2002) - PCA analysis of DAX and other papers
    - Just a few random factors drive the movement of the surace.
  - Brace et al (1998) - Stochastic implied volatility (market) models

- Problem with these approach is to ensure arbitrage-freeness of the implied volatility surface over time.

Construction Site - Dynamics and Hedging

- In discrete state models
  - How to we handle the evolution of time with these models?
  - Can we use them to hedge / superhedge a position / its risk?
- In continuous surface models
  - How can we ensure that the *dynamic* behaviour of the model is reasonable? Otherwise the hedge implied by any model is questionable.
    - E.g.: Local Volatility flattens out, but the market looks the “same” every day…
  - This is linked to the general question how we hedge in a model which has a certain set of parameters which are not stable in time.
    - Mean-Variance hedging etc can be applied once a model with stable parameters is found.
    - Risk-based hedging must be developped.
- Market-Models ?

Literature

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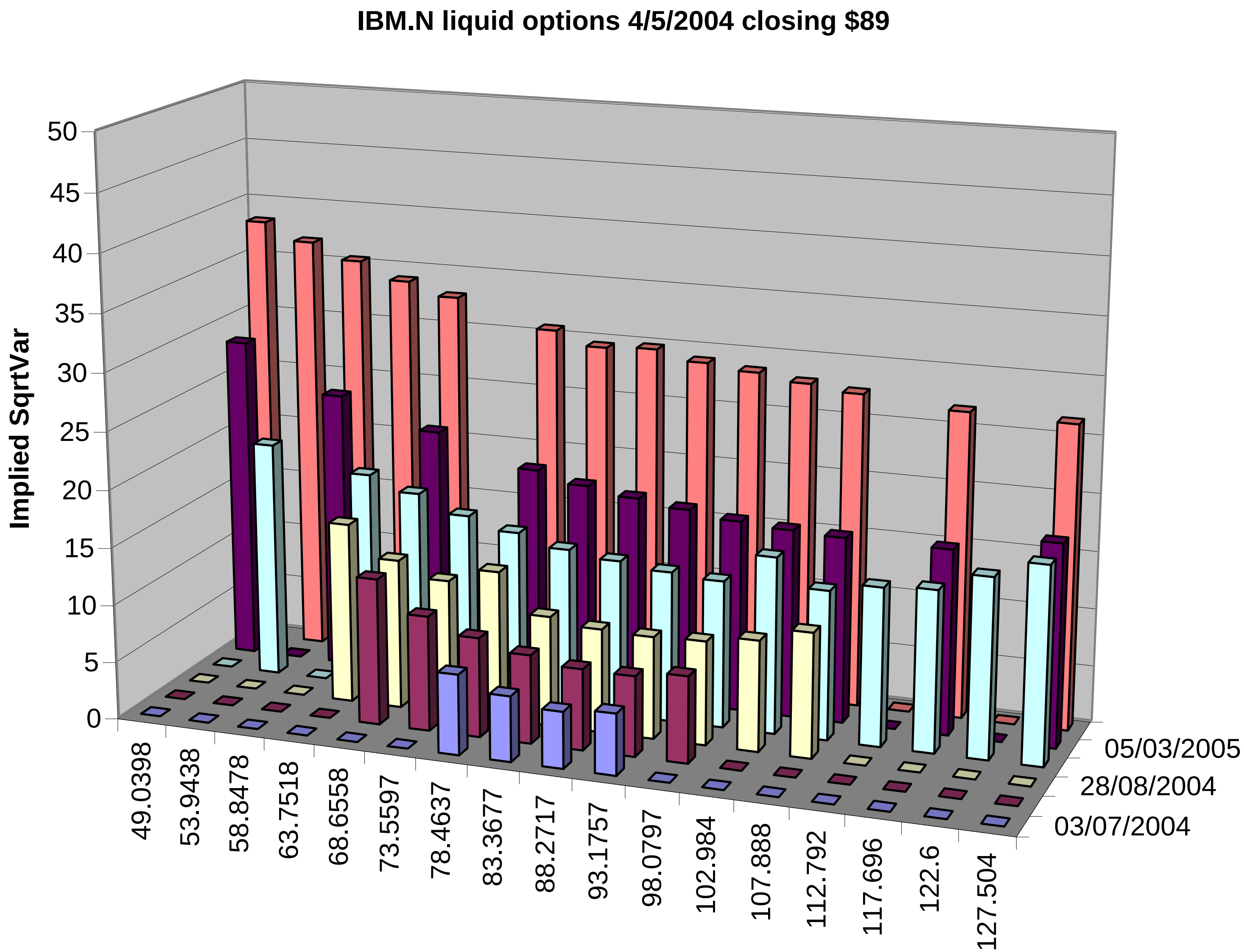
IBM.N liquid options 4/5/2004 closing $89
Implied SqrtVar
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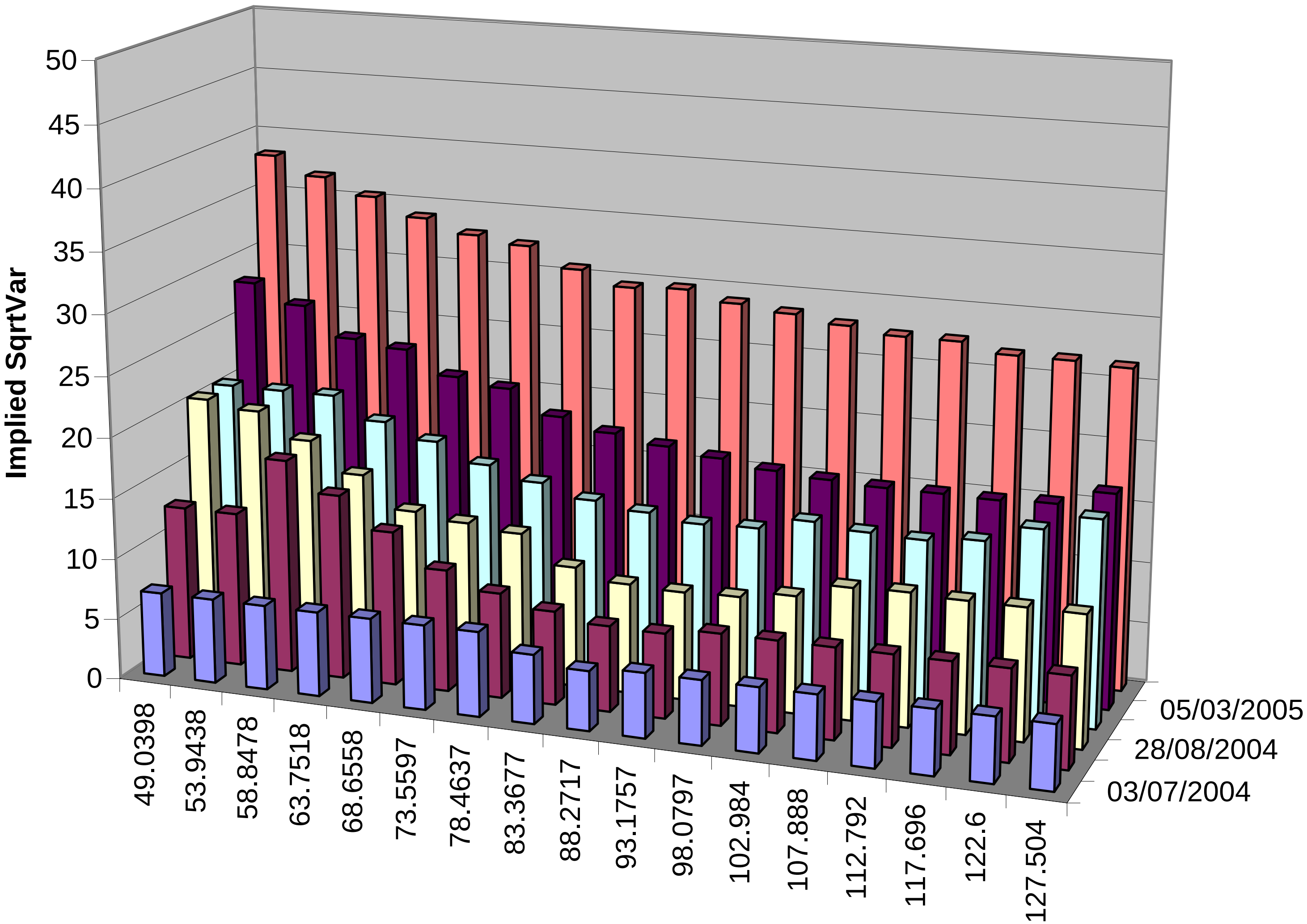
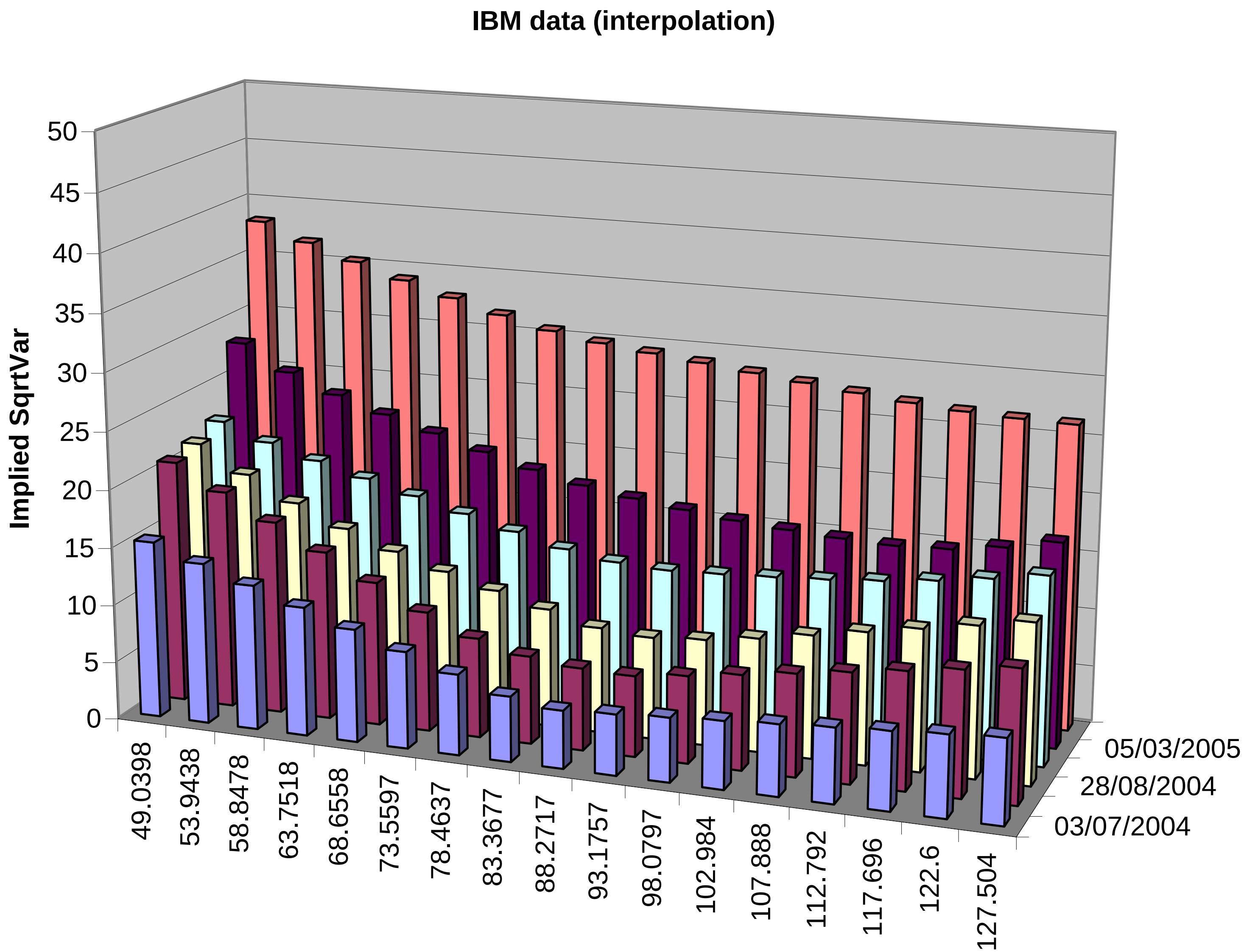
05/03/2005
28/08/2004
03/07/2004

IBM data (arbitrage-free)





IBM data (interpolation)
Implied SqrtVar
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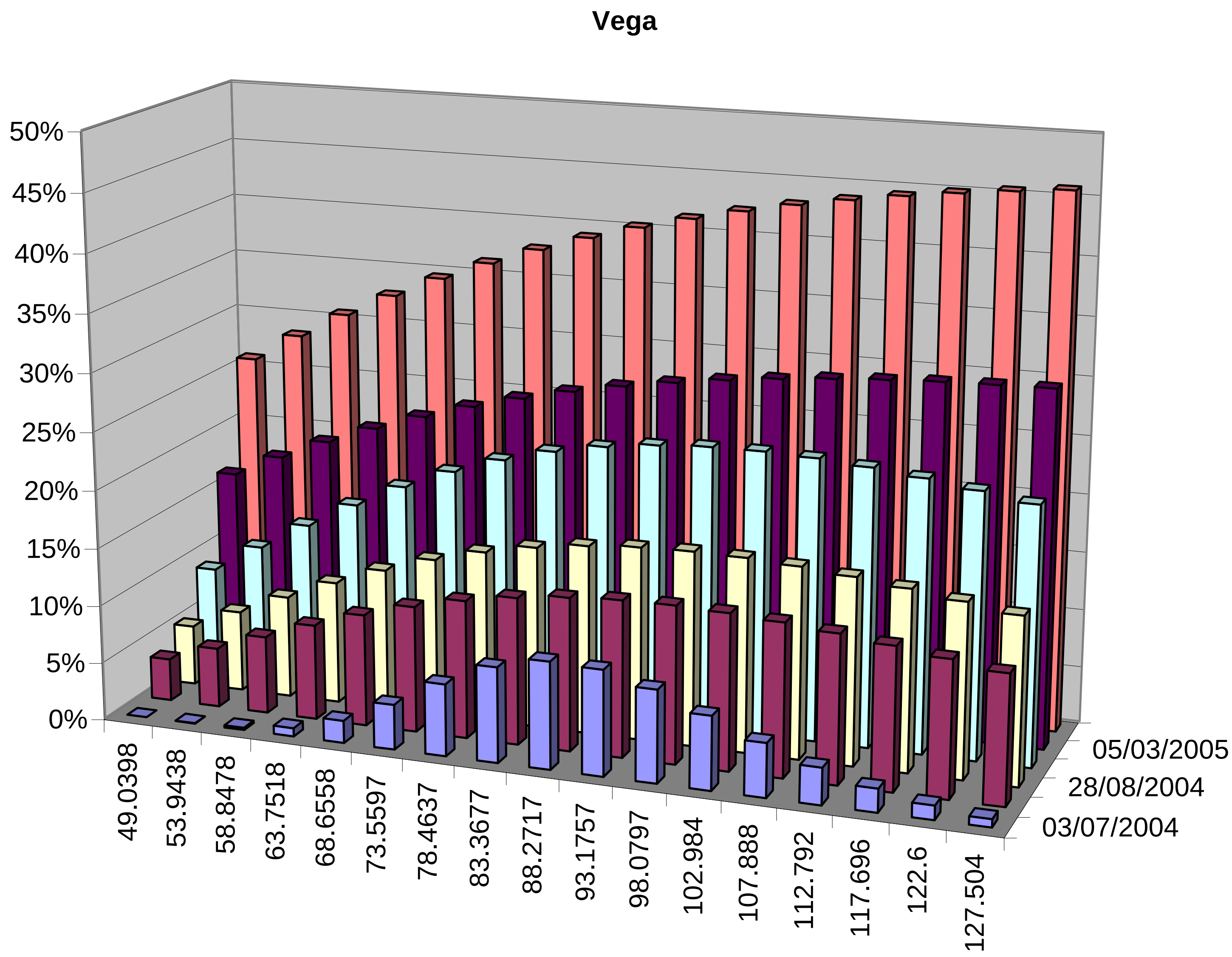

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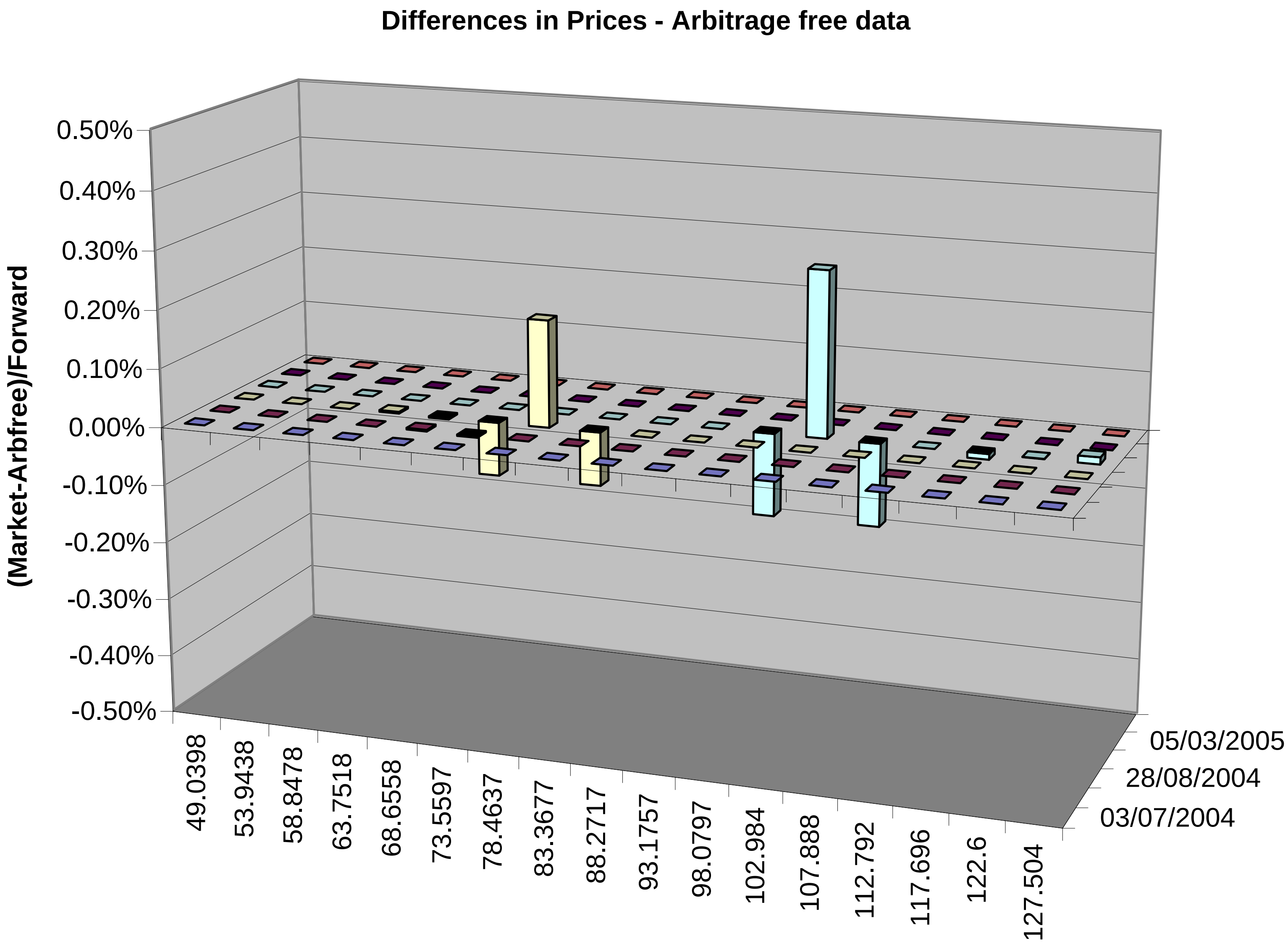
Differences in Prices - Arbitrage free data
(Market-Arbfree)/Forward
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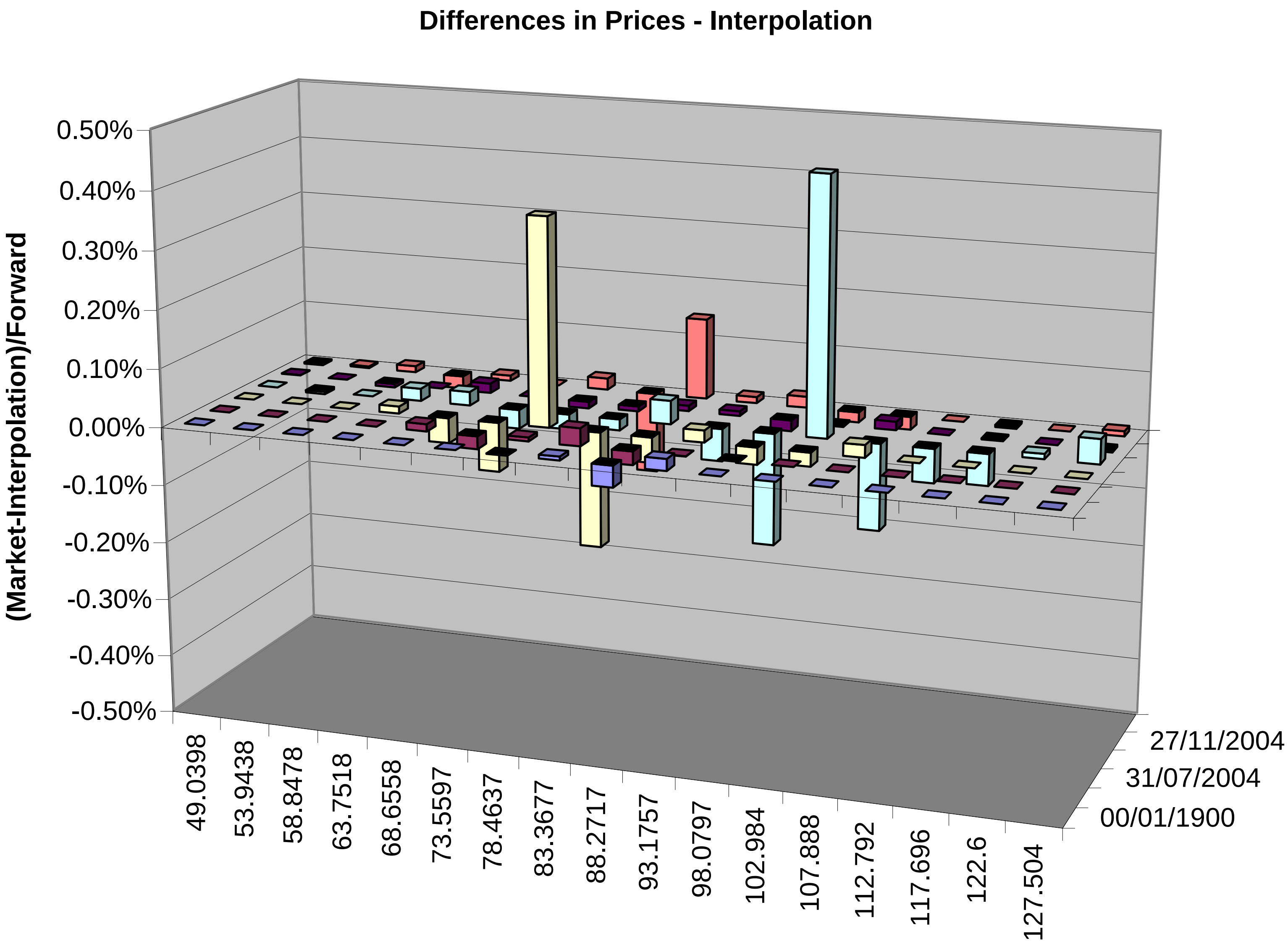

Differences in Prices - Interpolation
(Market-Interpolation)/Forward
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